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Superfield theories in tensorial superspaces and the dynamics of higher spin fields

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Abstract: We present the superfield generalization of free higher spin equations in tensorial superspaces and analyze tensorial supergravities with $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$ holonomy as a possible framework for the construction of a non-linear higher spin field theory. Surprisingly enough, we find that the most general solution of the supergravity constraints is given by a class of superconformally flat and $\mathrm{OSp}(1|n)$-related geometries. Because of the conformal symmetry of the supergravity constraints and of the higher spin field equations such geometries turn out to be trivial in the sense that they cannot generate a 'minimal' coupling of higher spin fields to their potentials even in curved backgrounds with a non-zero cosmological constant. This suggests that the construction of interacting higher spin theories in this framework might require an extension of the tensorial superspace with additional coordinates such as twistor-like spinor variables which are used to construct the $\mathrm{OSp}(1|2n)$ invariant ('preonic') superparticle action in tensorial superspace.

Keywords: M-Theory, Space-Time Symmetries, Supergravity Models, Superspaces.

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Contents

1. Introduction

The problem of self–consistent interactions of higher spin fields is one of the longstanding problems of theoretical physics (see [1] for references, [2] for an elementary review and [5] for recent progress). It is known that higher spin fields can consistently interact in a space–time with a non–vanishing cosmological constant. The gravitational interactions of the fermionic fields require the space–time to be of an anti-de-Sitter type [3, 4]. The interactions should simultaneously involve an infinite set of fields of an arbitrary high spin and their higher derivatives [6, 7, 3, 8, 4].

Several powerful methods have been developed to deal with theories which contain an infinite tower of higher spin fields. In particular, the star product formalism was used to construct higher spin theories [9, 10] even earlier than it was applied to the study of effects of non-commutativity in String Theory [11]. Actually, String Theory itself contains an infinite tower of interacting massive higher spin excitations. In a tensionless string limit the higher spin modes become massless and in a linear approximation satisfy free higher spin equations of motion (see e.g. [12] and references therein for more details).[1] However a much more non–trivial problem is to extract from the string effective action the information about the structure of higher spin interactions.

In [14] Fronsdal proposed another way of formulating higher-spin field theory. He conjectured that four-dimensional higher spin field theory can be realized as a field theory on a ten–dimensional 'tensorial' manifold parametrized by the coordinates

$$\begin{aligned} &X^{\alpha\beta} = X^{\beta\alpha} = \frac{1}{2}x^m\gamma_m^{\alpha\beta} + \frac{1}{4}y^{mn}\gamma_{mn}^{\alpha\beta}\,, \\ &m,n = 0,1,2,3\,; \qquad \alpha,\beta = 1,2,3,4\,, \end{aligned} \tag{1.1}$$

where x^m are associated with four coordinates of the conventional $D = 4$ space-time and six $y^{mn} = -y^{mn}$ describe spin degrees of freedom.

The assumption was that by analogy with, for example $D = 10$ or $D = 11$ supergravities, there may exist a theory in a ten–dimensional space whose alternative Kaluza-Klein reduction may lead in $D = 4$ to an infinite tower of fields with increasing spins instead of the infinite tower of Kaluza-Klein particles of increasing mass. It was argued that the symmetry group of the theory should be $\mathrm{OSp}(1|8) \supset \mathrm{SU}(2,2)$, which contains the $D = 4$ conformal group as a subgroup such that an irreducible (oscillator) representation of $\mathrm{OSp}(1|8)$ contains each and every massless higher spin representation of $\mathrm{SU}(2,2)$ only once. So the idea was that using a single representation of $\mathrm{OSp}(1|8)$ in the ten–dimensional tensorial space one could describe an infinite tower of higher spin fields in $D = 4$ space-time.

This proposal (rather accidentally) found its dynamical realization in the $\mathrm{OSp}(1|2n)$-invariant model of a twistor superparticle propagating in a flat tensorial superspace $(X^{\alpha\beta} = X^{\beta\alpha}, \theta^\alpha)$ $(\alpha, \beta = 1, \ldots, n$, with $n = 4$ corresponding to the Fronsdal case (1.1)) [15, 16]. The quantization of this model was shown [17] to produce the infinite tower of free massless fields of all possible spins in $D = 4$ space-time and an infinite set of higher spin fields in higher dimensions. In the general case the bosonic dimension of the tensorial superspace is $\frac{n(n+1)}{2}$. In particular, the case $n = 32$ has been considered (see [18]) as a point-like model for a BPS preon, a hypothetical constituent of M-theory [19].

[1] Note that these papers deal with a tensionless limit of ordinary (super)strings which differs from tensionless or so called null (super)strings [13].

The superparticle action in the flat tensorial superspace has the following form

$$S = \int d\tau \, [\dot{X}^{\alpha\beta}(\tau) - i\dot{\theta}^{\alpha}(\tau)\theta^{\beta}(\tau)]\lambda_\alpha\lambda_\beta \,, \tag{1.2}$$

where $\lambda_\alpha(\tau)$ are auxiliary commuting spinor variables. From (1.2) it follows that the particle momentum is $P_{\alpha\beta} = \lambda_\alpha\lambda_\beta$, which in the tensorial spaces associated with 4, 6 and 10-dimensional space-times implies that the quantum states of the superparticle are massless [15, 16]. Note that this is the direct analog and generalization of the Cartan-Penrose (twistor) realization of the light-like momentum of massless states.

The action (1.2) is non-manifestly invariant under the rigid transformations of OSp$(1|2n)$ [15, 16, 17, 21, 22, 24] but is manifestly invariant under the transformations of its subgroup GL(n) acting on X, θ and λ as follows

$$X'^{\alpha\beta} = X^{\alpha'\beta'} G_{\alpha'}{}^{\alpha} G_{\beta'}{}^{\beta} \,, \qquad \theta'^{\alpha} = \theta^{\alpha'} G_{\alpha'}{}^{\alpha} \,, \qquad \lambda'_\alpha = G^{-1\alpha'}_{\alpha} \lambda_{\alpha'} \,. \tag{1.3}$$

Superparticle models and free field theories in flat tensorial superspaces and on supergroup manifolds OSp$(1|n)$ have been studied in detail in [17] and [20]–[26]. It was conjectured in [20, 21] and shown in [23, 24] that a field theory on OSp$(1|4)$ is classically equivalent to the OSp$(1|8)$-invariant free higher spin field theory in AdS_4.

Interestingly enough, the spectrum of the quantum states and the wave equations which one obtains by quantizing the particle propagating in the *bosonic* tensorial space is *supersymmetric* and possesses OSp$(1|2n)$ symmetry [21], while the spectrum of the quantum states of the particle propagating in tensorial *super*space is the doubly degenerate spectrum of the 'bosonic' tensorial particle [17].

In the 'bosonic' case (i.e. when $\theta^\alpha = 0$) the quantization of the model (1.2) results in the following equation of motion of the particle wave function $\Phi(X^{\alpha\beta}, \lambda_\gamma)$ [17]

$$(\partial_{\alpha\beta} - i\lambda_\alpha\lambda_\beta)\Phi(X,\lambda) = 0 \,, \tag{1.4}$$

which may be called the "preonic" equation in the light of the conjecture of [19].

Upon the Fourier transform of $\Phi(X,\lambda)$ into $C(X, y^\alpha) = \int d^n\lambda \, e^{i\lambda_\alpha \, y^\alpha} \Phi(X,\lambda)$ the equation (1.4) takes the following equivalent form [21]

$$\left(\partial_{\alpha\beta} + i \frac{\partial}{\partial y^\alpha}\frac{\partial}{\partial y^\beta}\right) C(X,y) = 0 \,. \tag{1.5}$$

As was first shown in [21] the only dynamical fields among the components of the series expansion of $C(X,y) = b(X) + f_\alpha(X)\, y^\alpha + \sum_{n=2}^{\infty} C_{\alpha_1\cdots\alpha_n}(X)\, y^{\alpha_1}\cdots y^{\alpha_n}$ are the scalar field $b(X)$ and the spinor (or 'svector') field $f_\alpha(X)$ which, as a consequence of (1.5), satisfy the following equations of motion

$$(\partial_{\alpha\beta}\partial_{\gamma\delta} - \partial_{\alpha\gamma}\partial_{\beta\delta})\, b(X) := 2\partial_{\alpha[\beta}\partial_{\gamma]\delta}\, b(X) = 0 \,, \tag{1.6}$$

$$\partial_{\alpha\beta} f_\gamma(X) - \partial_{\alpha\gamma} f_\beta(X) := 2\partial_{\alpha[\beta} f_{\gamma]}(X) = 0 \,. \tag{1.7}$$

The equations (1.4)–(1.7) are OSp$(1|2n)$ invariant [21], the subgroup GL(n) of OSp$(1|2n)$ being a manifest symmetry of these equations. The fields $b(X)$ and $f_\alpha(X)$ are superpartners whose OSp$(1|2n)$ transformations the reader can find in [21]. Below we present only their part which corresponds to rigid supersymmetry and superconformal boosts with parameters

ϵ^α and s_α, respectively

$$\delta b(X) = \epsilon^\alpha\, f_\alpha(X) + 2 s_\alpha\, X^{\alpha\beta} f_\beta(X)\,, \qquad \delta f_\alpha(X) = \epsilon^\beta\, \partial_{\beta\alpha}\, b(X) + 2 s_\gamma X^{\gamma\beta}\, \partial_{\beta\alpha}\, b(X)\,. \quad (1.8)$$

In the case of $n = 4$ (1.1) the fields $b(X)$ and $f_\alpha(X)$ subject to eqs. (1.6) and (1.7) describe the infinite tower of the massless (and thus conformally invariant) fields of all possible integer and half-integer spins in the physical four-dimensional subspace of the ten-dimensional tensorial space [14, 21]. In the cases of $n = 8$ and $n = 16$ which correspond to $D = 6$ and $D = 10$ space-time, respectively, the equations (1.6) and (1.7) describe conformally invariant higher spin fields with self-dual field strengths (work in progress).

The field strengths of the $D = 4$ higher spin fields are components of the series expansion of $b(X) = b(x^l, y^{mn})$ and $f_\alpha(X) = f_\alpha(x^l, y^{mn})$ in powers of the tensorial coordinate y^{mn}

$$\begin{aligned}
b(x^l,\, y^{mn}) &= \phi(x) + y^{m_1 n_1} F_{m_1 n_1}(x) + \\
&\quad + y^{m_1 n_1}\, y^{m_2 n_2} \left[C_{m_1 n_1, m_2 n_2}(x) + \frac{1}{4} \partial_{[n_1}\, \eta_{m_1][m_2}\, \partial_{n_2]} \phi(x) \right] + \\
&\quad + \sum_{s=3}^{\infty} y^{m_1 n_1} \cdots y^{m_s n_s}\, [C_{m_1 n_1, \cdots, m_s n_s}(x) + \cdots]\,, \\
f^\alpha(x^l, y^{mn}) &= \psi^\alpha(x) + y^{m_1 n_1} \left[\Psi^\alpha_{m_1 n_1}(x) + \frac{1}{2} \partial_{m_1} (\gamma_{n_1} \psi)^\alpha \right] + \\
&\quad + \sum_{s=\frac{5}{2}}^{\infty} y^{m_1 n_1} \cdots y^{m_{s-\frac{1}{2}} n_{s-\frac{1}{2}}} \left[\Psi^\alpha_{m_1 n_1, \cdots, m_{s-\frac{1}{2}} n_{s-\frac{1}{2}}}(x) + \cdots \right].
\end{aligned} \quad (1.9)$$

In (1.9) $\phi(x)$ and $\psi^\alpha(x)$ are scalar and spin 1/2 field, $F_{m_1 n_1}(x)$ is the Maxwell field strength, $C_{m_1 n_1, m_2 n_2}(x)$ is the Weyl curvature tensor of the linearized gravity, $\Psi^\alpha_{m_1 n_1}(x)$ is the Rarita-Schwinger field strength and other terms in the series stand for strengths of spin-s fields (which also contain contributions of derivatives of lower spin fields denoted by dots, as in the case of the Weyl curvature and of the Rarita-Schwinger field).

The $\mathrm{OSp}(1|2n)$ invariant equations of motion of the fields $b(X)$ and $f_\alpha(X)$ propagating on the group manifold $\mathrm{Sp}(n)$ (see eqs. (2.12) and (2.13) of section 2.2) have been derived in [25] from the $\mathrm{Sp}(n)$ counterpart of (1.4) and (1.5)

$$\left[\nabla_{\alpha\beta} - \frac{i}{2} (Y_\alpha Y_\beta + Y_\beta Y_\alpha) \right] C(X, y) = 0\,, \qquad Y_\alpha \equiv i \frac{\partial}{\partial y^\alpha} + \frac{\varsigma}{4} y_\alpha\,, \quad (1.10)$$

where $\nabla_{\alpha\beta}$ are covariant derivatives generating the algebra $\mathrm{Sp}(n)$

$$[\nabla_{\alpha\beta}, \nabla_{\gamma\delta}] = \varsigma C_{\alpha(\gamma} \nabla_{\delta)\beta} + \varsigma C_{\beta(\gamma} \nabla_{\delta)\alpha}\,, \quad (1.11)$$

$C_{\alpha\beta}$ is a simplectic metric and ς is a constant proportional to the inverse AdS radius (the square root of the cosmological constant). In [23] the general solution of the equations (1.10) and their generalization to the supergroup manifolds $\mathrm{OSp}(N|n)$ were constructed and analyzed. For instance, in the case of $n = 4$ the equations (1.10) are equivalent to an infinite system of equations of motion of all the integer and half-integer higher spin fields propagating in AdS_4 [23, 25].

At this point we should make the following comment. In the formulation described in eqs. (1.4)–(1.11) the fields $b(X)$ and $f_\alpha(X)$ have the same statistics, namely they are Grassmann even if $\Phi(X, \lambda)$ or $C(X, y)$ is Grassmann even, while if we would like $b(X)$ and $f_\alpha(X)$ to form a scalar $\mathrm{OSp}(1|n)$ supermultiplet we should assign to $f_\alpha(X)$ the fermionic statistics. An a priori un-physical statistics of a part of higher spin fields is a generic feature of the unfolded formulations of higher spin field theory involving twistor-like Grassmann even spinor variables λ_α or y^α [27, 21]. To single out the fields with physically correct statistics one can use several equivalent prescriptions [28, 10, 21]. In our case the most appropriate one is the following 'parity conservation' requirement used in [17, 1]. One should consider the complete (doubly degenerate) spectrum of states of the quantum superparticle model (1.2) which on the mass shell is described by a generic Grassmann even superfield of the form

$$g_0(X, \lambda, \theta) = \Phi(X, \lambda) + i(\lambda_\alpha \theta^\alpha)\, \Psi(X, \lambda)\,, \tag{1.12}$$

where $\Psi(X, \lambda)$ is a Grassmann odd counterpart of $\Phi(X, \lambda)$. In $\Psi(X, \lambda)$ the half integer spin fields have the correct statistics while the integer spin fields do not. We now notice that the Grassmann parity of $g_0(X, \lambda, \theta)$ can be related to the parity of $g_0(X, \lambda, \theta)$ under the change of the sign of λ ($\lambda \to -\lambda$). If $g_0(X, \lambda, \theta) = g_0(X, -\lambda, \theta)$ then $\Phi(X, \lambda)$ and $\Psi(X, \lambda)$ are expanded in the integer and half integer series of λ, respectively, and contain the fields of only physically appropriate statistics (see [17] for details). Thus, strictly speaking, one should regard the fields $b(X)$ and $f_\alpha(X)$ of eqs. (1.6), (1.7) and (2.12), (2.13) as ones which come, respectively, from the field $\Phi(X, \lambda)$ and $\Psi(X, \lambda)$ of (1.12), namely $b(X) = \int d^n\lambda\, \Phi(X, \lambda)$ and $f_\alpha(X) = \int d^n\lambda\, \lambda_\alpha \Psi(X, \lambda)$. We shall discuss this in more detail in section 2.

Since the equations (1.6), (1.7) and their AdS counterparts are supersymmetric, a natural question arises whether these equations can be formulated as superfield equations in a corresponding tensorial superspace[2] and whether they allow for a nonlinear generalization which would result in an interacting theory of higher spin fields. In this paper we study these problems.

First we combine the scalar field $b(X)$ and the spinor field $f_\alpha(X)$ into a scalar superfield $\Phi(X, \theta)$ and find simple superfield equations for $\Phi(X, \theta)$ which reproduce (1.6), (1.7) and the "preonic" equation (1.4). Then we look for a non-linear generalization of the superfield equations. Our initial assumption has been that a class of non-linear models of this kind can be constructed in a consistent geometrical way by considering a supergravity in tensorial superspace. A stronger conjecture might be that the tensorial supergravity itself is an example of a theory of interacting higher spin fields. If it was so, the superdiffeomorphisms and the local $\mathrm{GL}(n)$ or $\mathrm{SL}(n)$ structure group transformations of the tensorial superspace could generate infinite higher spin superalgebras in ordinary space-time.

[2] A superfield formulation of the unfolded higher spin dynamics [27, 21] was considered in [29].

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Our reasoning behind the idea to look for a non-linear dynamics of higher spin fields within a superfield formulation of tensorial supergravity and not, for instance within a bosonic tensorial gravity has been two fold

- the superfield equations of motion of the free higher spin fields are much simpler than their component counterparts and hence may be more appropriate for a non-linear generalization and
- as the experience of dealing with conventional superfield gauge and supergravity theories teaches us, the imposition of constraints on the superfield contents of these theories reduces the number of possible choices and in many cases produces a complete set of superfield equations of motion whose form would be otherwise hard to guess in the absence of clear group-theoretical and geometrical guidelines.

As we shall see, the supergeometry of the tensorial supergravity with $\mathrm{GL}(n)$ or $\mathrm{SL}(n)$ holonomy which we derive from the requirement of the κ-invariance of the superparticle action in the curved superspace background resembles that of $N=1$, $D=2$ and $D=3$ supergravity. We find that general solutions of the supergravity constraints are tensorial superspaces conformally related to flat tensorial superspace or to the supergroup manifold $\mathrm{OSp}(1|n)$. Because of the conformal symmetry of the supergravity constraints and of the scalar superfield equation such a geometry is trivial in the sense that it cannot generate a kind of 'minimal' coupling of higher spin fields to their potentials. So our expectations to find non-linear higher spin field equations in the framework of tensorial supergravity have not been materialized yet. However, we believe that the results obtained lay a geometrical basis for a new class of models formulated in tensorial superspaces and may be useful for further development of this subject in various directions. One of them may hopefully bring us to a non-linear higher spin dynamics.

The paper is organized as follows. In section 2 we construct the equations of motion of a scalar superfield $\Phi(X,\theta)$ in the flat tensorial superspace and on the supergroup manifold $\mathrm{OSp}(1|n)$. We also find a superfield generalization of the "preonic" equation (1.4) and of its AdS counterpart (1.10).

In section 3 we introduce the supergeometry of a curved tensorial superspace with the holonomy group $\mathrm{GL}(n)$ and find constraints on its torsion and curvature which are required by the κ-invariance of the ('preonic') superparticle action. We then impose additional conventional supergravity constraints and study the consistency of the torsion and curvature Bianchi identities. In particular we find that, as in the case of $N=1$, $D=3$ supergravity [30], the supergeometry with $\mathrm{SL}(n)$ holonomy is described by an antisymmetric tensor superfield $R_{\alpha\beta}(X,\theta)$ and by a totally symmetric field $G_{\alpha\beta\gamma}(X,\theta)$.

Section 4 is devoted to the consideration of the dynamics of the scalar superfield in an external tensorial supergravity background. It is shown that its consistency requires the background supergeometry to have $\mathrm{SL}(n)$ holonomy.

In section 5 we describe generalized Weyl (superconformal) transformations of the supervielbeins and superconnection which leave the constraints form-invariant and study superconformally flat and OSp-related geometries of tensorial superspaces.

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In section 6 we show that (being superconformally invariant) the dynamics of the scalar superfield propagating in a conformally flat or OSp$(1|n)$-related tensorial superspace is described by the free scalar superfield equation in flat superspace or on the supergroup manifold OSp$(1|n)$ and hence does not lead to a non-trivial interacting theory of higher spin fields.

The general solution of the tensorial supergravity constraints is considered in section 7. It is shown that (up to possible topological subtleties) the conformal tensorial superspaces are the only solutions of this theory.

In conclusion we summarize the main results obtained and discuss possible ways in which they can be developed.

2. Superfield generalization of the massless higher spin equations

2.1 Scalar superfield equations in flat tensorial superspace

Let us consider a scalar superfield

$$\Phi(X^{\alpha\beta},\theta^{\gamma}) = b(X) + f_{\alpha}(X)\,\theta^{\alpha} + \sum_{i=2}^{n}\phi_{\alpha_1\cdots\alpha_i}(X)\,\theta^{\alpha_1}\cdots\theta^{\alpha_i} \tag{2.1}$$

in a flat tensorial superspace whose coordinates transform under rigid supertranslations as follows

$$\delta\theta^{\alpha} = \epsilon^{\alpha}\,, \qquad \delta X^{\alpha\beta} = \frac{i}{2}(\theta^{\alpha}\,\epsilon^{\beta} + \theta^{\beta}\,\epsilon^{\alpha}) = i\theta^{(\alpha}\,\epsilon^{\beta)}\,. \tag{2.2}$$

We are looking for a superfield equation for $\Phi(X,\theta)$ which would reproduce the equations (1.6) and (1.7) for the leading components of $\Phi(X,\theta)$ and from which it would follow that the higher components of the superfield $\Phi(X,\theta)$ are completely auxiliary and vanish on the mass shell. Since (1.6) and (1.7) are manifestly GL(n) invariant, the corresponding superfield equation should also possess this symmetry. Taking this into account we find that the only possible superfield equation quadratic in supercovariant derivatives $D_{\alpha} = \partial/\partial\theta^{\alpha} + i\theta^{\beta}\partial_{\beta\alpha}$ ($\{D_{\alpha},D_{\beta}\} = 2i\partial_{\alpha\beta}$) is

$$D_{[\alpha}D_{\beta]}\Phi(X,\theta) = 0\,. \tag{2.3}$$

It can be regarded as a generalization to the tensorial superspace of the defining conditions of a tensor supermultiplet in $D=4$ or of the equations of motion of a scalar supermultiplet in $D=3$.

The analysis of the equation (2.3) in flat tensorial superspace with an arbitrary even number n of the Grassmann coordinates shows that all components of the superfield (2.1) subject to (2.3) vanish, except for $b(X)$ and $f_{\alpha}(X)$, and the latter satisfy the equations (1.6) and (1.7).

The equation (2.3) can be derived in a rigorous way from a superfield equation which one gets by quantizing the tensorial superparticle model (1.2). As was considered in detail in [17] the quantum states of the tensorial superparticle form a bosonic superfield

$$\Upsilon(X,\theta,\lambda,\chi) = g_0(X,\theta,\lambda) + i\chi\,g_1(X,\theta,\lambda)\,, \tag{2.4}$$

where χ is a real single Clifford variable ($\chi^2 = 1$) of the Grassmann odd parity. As has been mentioned in the Introduction, to have the correct relation between spin and statistics of the components of the series expansion of g_0 and g_1 in powers of λ_α, we require that (2.4) is an even function under the change of sign of λ and χ ($\lambda \to -\lambda,\ \chi \to -\chi$), namely $\Upsilon(X,\theta,\lambda,\chi) = \Upsilon(X,\theta,-\lambda,-\chi)$. This implies that $g_0(X,\theta,\lambda) = g_0(X,\theta,-\lambda)$ and $g_1(X,\theta,\lambda) = -g_1(X,\theta,-\lambda)$.

The superfield (2.4) satisfies the first order differential equation [17]

$$(D_\alpha - \chi\,\lambda_\alpha)\Upsilon(X,\theta,\lambda,\chi) = 0\,. \tag{2.5}$$

From (2.5) it follows that

$$D_\alpha\, g_0 - i\,\lambda_\alpha\, g_1 = 0\,, \qquad D_\alpha\, g_1 - i\,\lambda_\alpha\, g_0 = 0\,. \tag{2.6}$$

Hence, for example g_1 can be expressed in terms of $D_\alpha\, g_0$

$$g_1 = -i\mu^\alpha\, D_\alpha\, g_0\,, \tag{2.7}$$

where μ^α is "inverse" of λ_α in the sense that $\mu^\alpha\lambda_\alpha = 1$.

Thus only one superfield component of (2.4), e.g. $g_0(X,\theta,\lambda) = g_0(X,\theta,-\lambda)$, is independent. Now taking the derivative D_α of (2.6) we find that g_0 should obey the equation

$$(D_\alpha D_\beta + \lambda_\alpha\lambda_\beta)\, g_0(X,\theta,\lambda) = 0\,. \tag{2.8}$$

The symmetric part of (2.8) is

$$(\partial_{\alpha\beta} - i\lambda_\alpha\,\lambda_\beta)\, g_0(X,\theta,\lambda) = 0\,,$$

which is similar to (1.4), while the antisymmetric part is

$$D_{[\alpha}D_{\beta]}\, g_0(X,\theta,\lambda) = 0\,. \tag{2.9}$$

Thus, we can regard (2.8) and/or (2.5) as a superfield generalization of the "preonic" equation (1.4).

Integrating (2.9) over λ and defining $\Phi(X,\theta) = \int d^n\lambda\, g_0(X,\theta,\lambda)$, so that $b(X) = \int d^n\lambda\, g_0(X,\theta,\lambda)|_{\theta=0}$ and $f_\alpha(X) = \int d^n\lambda\, D_\alpha\, g_0(X,\theta,\lambda)|_{\theta=0}$, we get the equation (2.3). Thus, on the mass shell the scalar superfield (2.1) is linear in θ^α, which is in accordance with the form of the wave function describing on-shell quantum states of the tensorial superparticle discussed in the Introduction (eq. (1.12)).

2.2 Scalar superfield equations on OSp$(1|n)$

Let us now consider the case when $X^{\alpha\beta}$ and θ^α parametrize a supergroup manifold OSp$(1|n)$ and find the corresponding generalization of the superfield equation (2.3). For this we should replace the flat covariant derivatives D_α with OSp$(1|n)$ covariant derivatives ∇_α which extend the $sp(n)$ algebra (1.11) to the $osp(1|n)$ superalgebra[3]

$$\{\nabla_\alpha, \nabla_\beta\} = 2i\nabla_{\alpha\beta}\,, \qquad [\nabla_{\alpha\alpha'}, \nabla_\beta] = \varsigma\, C_{\beta(\alpha}\nabla_{\alpha')}\,. \tag{2.10}$$

[3]Explicit expressions for the OSp$(1|n)$ Cartan forms and covariant derivatives in particular parametrizations has been given in [20, 25] and for the OSp$(N|n)$ Cartan forms in a generic parametrization in [23].

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The scalar superfield equation on the supergroup manifold $\mathrm{OSp}(1|n)$ has the following form

$$\left(\nabla_{[\alpha}\nabla_{\beta]} + i\frac{\varsigma}{4}C_{\alpha\beta}\right)\Phi(X,\theta) = 0\,. \tag{2.11}$$

The equation (2.11) reduces to the following equations on the dynamical components of $\Phi(X,\theta)$ [25]

$$\nabla_{\alpha[\beta}\nabla_{\gamma]\delta}b(X) = \frac{\varsigma}{4}\left(C_{\alpha[\beta}\nabla_{\gamma]\delta}+C_{\delta[\beta}\nabla_{\gamma]\alpha}-C_{\beta\gamma}\nabla_{\alpha\delta}\right)b(X)+\frac{\varsigma^2}{16}\left(C_{\alpha\delta}C_{\beta\gamma}-C_{\alpha[\beta}C_{\gamma]\delta}\right)b(X)\,, \tag{2.12}$$

$$\nabla_{\alpha[\beta}f_{\gamma]}(X) = -\frac{\varsigma}{4}\left(C_{\alpha[\gamma}f_{\beta]}(X) + C_{\beta\gamma}f_{\alpha}(X)\right). \tag{2.13}$$

The coefficient in front of the second term of (2.11) is fixed by checking the integrability of this equation. To this end we observe that $\nabla_{\alpha[\beta}\nabla_{\gamma]\delta}b(X) = (\nabla_{\alpha[\beta}\nabla_{\gamma]\delta}\Phi(X,\theta))|_{\theta=0}$ and hence in view of (1.11)

$$\begin{aligned}\nabla_{(\alpha[\beta}\nabla_{\gamma]\delta)}\,b(X) &= \frac{1}{2}(\nabla_{\alpha[\beta}\nabla_{\gamma]\delta} + \nabla_{\delta[\beta}\nabla_{\gamma]\alpha})b(X) = \frac{1}{2}[\nabla_{\alpha[\beta},\nabla_{\gamma]\delta}]\,b(X)\\ &= \frac{\varsigma}{4}\left(C_{\alpha[\beta}\nabla_{\gamma]\delta}\,b(X) + C_{\delta[\beta}\nabla_{\gamma]\alpha}\,b(X) - C_{\beta\gamma}\nabla_{\alpha\delta}\,b(X)\right).\end{aligned} \tag{2.14}$$

The equation (2.14) is then compared with the bosonic equation (2.12) which follows from (2.11). This fixes in the latter the factor $\frac{i\varsigma}{4}$.

A superfield generalization of the "AdS preonic" equation (1.10) considered in [23] is

$$(\nabla_\alpha - \chi Y_\alpha)\Upsilon(X,\theta,\lambda,\chi) = 0\,, \tag{2.15}$$

while the $\mathrm{OSp}(1|N)$ analog of eq. (2.8) is

$$(\nabla_\alpha\nabla_\beta + Y_\beta Y_\alpha)\,g_0(X,\theta,\lambda) = 0\,, \qquad Y_\alpha = \lambda_\alpha - \frac{i\varsigma}{4}\,C_{\alpha\beta}\,\frac{\partial}{\partial\lambda_\beta}\,. \tag{2.16}$$

We observe that the superfield equations (2.3) and (2.11) are much simpler than their component counterparts and therefore it is natural to take them as a starting point in the search for a non-linear generalization of the higher-spin field equations. Since the scalar field contains only the linearized field strengths of the higher spin fields one needs to find a room for higher spin field potentials which are required for the construction of 'minimal' higher spin interactions. In this respect on can consider supergravity in tensorial superspace and its coupling to the scalar superfield as a model which might provide us with minimal-like higher spin interactions via supervielbeins and superconnections.

3. Geometry of tensorial superspace

3.1 The definition of tensorial supergeometry

As in the conventional supergravity case, curved tensorial superspace geometry is described by the supervielbein one forms $E^{\alpha\beta}(Z) = E^{\beta\alpha}(Z) = dZ^M E_M{}^{\alpha\beta}(Z)$ and $E^\alpha(Z) = dZ^M E_M{}^\alpha(Z)$. The supercoordinates $Z^M = (X^{\mu\nu},\theta^\rho)$ are assumed to transform under the superdiffemorphisms $Z'^M = f^M(Z^N)$ $(\mathrm{sdet}(\partial f^M/\partial Z^N) \neq 0)$ which leave the supervielbeins invariant $(E'^{\mathcal{A}}(Z') = E^{\mathcal{A}}(Z))$.

We have seen that in the flat case the superparticle model (1.2) is manifestly invariant under the rigid transformations of the group GL(n) (1.3), which can be regarded as a kind of the "Lorentz" group in the tensorial superspace. We shall therefore assume that in the tensorial supergravity GL(n) plays the role of a generalized local Lorentz group acting in the co-tangent tensorial superspace whose local basis is given by the supervielbeins $E^{\mathcal{A}} = (E^{\alpha\beta}, E^{\gamma})$. As so, by analogy with the conventional spin connection of general relativity and supergravity we introduce the GL(n) connection

$$\Omega_\beta{}^\alpha := dZ^M \Omega_{M\beta}{}^\alpha \equiv E^{\mathcal{A}} \Omega_{\mathcal{A}\beta}{}^\alpha \,, \tag{3.1}$$

the torsion 2-forms (where $\mathcal{D}$ stands for the GL(n)-covariant differential)

$$T^{\alpha\beta} := \mathcal{D}E^{\alpha\beta} \equiv dE^{\alpha\beta} - E^{\alpha\gamma} \wedge \Omega_\gamma{}^\beta - E^{\beta\gamma} \wedge \Omega_\gamma{}^\alpha \,, \tag{3.2}$$

$$T^{\alpha} := \mathcal{D}E^{\alpha} \equiv dE^{\alpha} - E^{\beta} \wedge \Omega_\beta{}^\alpha \,, \tag{3.3}$$

and the curvature of the GL(n) connection

$$\mathcal{R}_\beta{}^\alpha := d\,\Omega_\beta{}^\alpha - \Omega_\beta{}^\gamma \wedge \Omega_\gamma{}^\alpha \,. \tag{3.4}$$

The Ricci identity $\mathcal{D}\mathcal{D} = \mathcal{R}$ in our notation implies $\mathcal{D}\mathcal{D}E^\alpha \equiv -E^\beta \wedge \mathcal{R}_\beta{}^\alpha$.

In what follows we shall also discuss consequences of the restriction of the GL(n) curvature to the SL(n) curvature by imposing the tracelessness constraint $\mathcal{R}_\alpha{}^\alpha = 0$.

The next step is to find the constraints on tensorial supergeometry. In the case of conventional super Yang-Mills and supergravity theories there are different geometrical and physical guiding lines to get superfield constraints. The one which we have at our disposal is the κ-symmetry of the massless superparticle.

3.2 The massless superparticle in curved tensorial superspace

Let us consider the dynamics of a superparticle in a curved tensorial superspace and find restrictions on its supergeometry which follow from the requirement for the model to possess the same symmetries as in the flat limit. Thus, we shall derive the constraints on torsion and curvature of a supergravity in tensorial superspace using the conventional requirement that a superparticle or a superbrane propagating in the supergravity background should be invariant under κ-symmetry, as in the flat case.

3.2.1 Superparticle action, κ-symmetry and the basic torsion constraint in tensorial superspace

A straightforward generalization of the action (1.2) to the curved tensorial superspace is

$$S = \frac{1}{2}\int E^{\alpha\beta}(Z(\tau))\,\lambda_\alpha(\tau)\,\lambda_\beta(\tau) \;= \frac{1}{2}\int d\tau E_\tau^{\alpha\beta}\lambda_\alpha\lambda_\beta \,, \tag{3.5}$$

where the flat superform $dX^{\alpha\beta}(\tau) - id\theta^{(\alpha}\theta^{\beta)}(\tau)$ of eq. (1.2) has been replaced with the pull-back on the superparticle worldline of the bosonic supervielbein form $E^{\alpha\beta}(Z)$

$$E^{\alpha\beta}(Z(\tau)) := d\tau E_\tau^{\alpha\beta} = dZ^M(\tau) E_M^{\alpha\beta}(Z(\tau)) \,. \tag{3.6}$$

In the flat case the action (1.2) is invariant under local κ-symmetry with $n-1$ independent parameters, which means that the superparticle under consideration can be associated with a BPS state (called the BPS preon [19]) which preserves all but one supersymmetry [15]. The κ-symmetry transformations of the fields in the action (1.2) are

$$\delta_\kappa X^{\alpha\beta}(\tau) = i\delta_\kappa \theta^{(\alpha}(\tau)\,\theta^{\beta)}(\tau)\,, \quad \delta_\kappa\,\lambda_\alpha(\tau) = 0\,, \tag{3.7}$$

$$\delta_\kappa \theta^\alpha(\tau) = \sum_{I=1}^{n-1} \kappa^I(\tau)\,\mu_I^\alpha(\tau)\,, \tag{3.8}$$

where $\kappa^I(\tau)$ are $n-1$ fermionic parameters and $\mu_I^\alpha(\tau)$ is a set of $(n-1)$ auxiliary *bosonic* $\mathrm{GL}(n)$ vectors (or spinors of an $\mathrm{SO}(t, D-t) \subset \mathrm{GL}(n)$ for $n = 2^k$) which are orthogonal to $\lambda_\alpha(\tau)$[4],

$$\mu_I^\alpha(\tau)\lambda_\alpha(\tau) = 0\,, \qquad I = 1, \dots (n-1)\,. \tag{3.9}$$

Actually (3.8) describes the general solution of the equation

$$\delta_\kappa\theta^\alpha(\tau)\lambda_\alpha(\tau) = 0\,, \tag{3.10}$$

which can be used instead of (3.8) as the definition of $(n-1)$ parametric κ-symmetry.

The flat superspace action (1.2) is also invariant under the $n(n-1)/2$ parametric bosonic b-symmetry [15, 17], which can be treated as a bosonic 'superpartner' of the κ-symmetry,

$$\delta_b X^{\alpha\alpha'} = \mu_I^\alpha \mu_J^{\alpha'} b^{IJ}(\tau)\ (\Leftrightarrow\ \delta_b X^{\alpha\alpha'}\lambda_{\alpha'} = 0)\,, \qquad \delta_b\theta^\alpha(\tau) = 0\,, \qquad \delta_b\lambda_\alpha(\tau) = 0\,. \tag{3.11}$$

We would like the κ-symmetry as well as the b-symmetry to be also preserved in the supergravity background (see [37]). Such a requirement has a deep physical meaning: it implies that the limit of flat superspace (when the background fields tends to zero) is smooth and, in particular, that the number of the degrees of freedom of the dynamical system does not change in such a limit. The curved superspace generalization of the κ-symmetry and of the b-symmetry transformations (3.7) and (3.11) of the coordinate functions are, respectively,

$$i_\kappa E^{\alpha\alpha'} := \delta_\kappa Z^M E_M^{\alpha\alpha'} = 0\,, \quad i_\kappa E^\alpha := \delta_\kappa Z^M E_M^\alpha = \mu_I^\alpha \kappa^I(\tau)\,, \tag{3.12}$$

and

$$i_b E^{\alpha\alpha'} := \delta_b Z^M E_M^{\alpha\alpha'} = \mu_I^\alpha\mu_J^{\alpha'} b^{IJ}(\tau)\,, \qquad i_b E^\alpha := \delta_b Z^M E_M^\alpha = 0\,. \tag{3.13}$$

The variation of the bosonic spinor field $\lambda_\alpha(\tau)$, $\delta_\kappa\lambda_\alpha$ and $\delta_b\lambda_\alpha$ are to be defined from the invariance of the action.

The invariance of the action (3.5) under the κ- and b-transformations (3.12) and (3.13) requires the bosonic torsion of the tensorial superspace to be restricted by the constraints[5]

$$T^{\alpha\beta} = -iE^\alpha \wedge E^\beta + 2E^\gamma \wedge E^{\delta(\alpha} t_{\gamma\delta}{}^{\beta)}(Z) + E^{\gamma\gamma'} \wedge E^{\delta(\alpha}\, t_{\gamma\gamma'\,\delta}{}^{\beta)}(Z)\,. \tag{3.14}$$

[4] The bosonic spinors μ_I^α can be considered [18] as counterparts of the Killing spinors corresponding to an $(n-1)/n$ supersymmetric (BPS preon) solution of supergravity equations, which is still hypothetical for the standard supergravity but which exists in a Chern-Simons like supergravity [18].

[5] We should note that the requirement of the κ-symmetry itself already leads to the constraints (3.14), while taking into the consideration of the b-invariance makes the analysis simpler.

The complete set of the κ-symmetry and b-symmetry transformations leaving the action (3.5) invariant in the background (3.14) is

$$\begin{aligned} i_\kappa E^{\alpha\alpha'} &= 0\,, \qquad i_\kappa E^\alpha \lambda_\alpha = 0 \ \left(\Leftrightarrow\ i_\kappa E^\alpha = \mu^\alpha_I \kappa^I(\tau)\ \right), \\ \delta_\kappa \lambda_\alpha &= i_\kappa E^\beta t_{\beta\ \alpha}{}^\gamma \lambda_\gamma\,; \end{aligned} \tag{3.15}$$

$$\begin{aligned} i_b E^{\alpha\beta}\lambda_\beta = 0 \ (\Leftrightarrow\ i_b E^{\alpha\alpha'} &= \mu^\alpha_I \mu^{\alpha'}_J b^{IJ}(\tau)\ )\,, \qquad i_b E^\alpha = 0\,, \\ \delta_b \lambda_\alpha &= \frac{1}{2} i_b E^{\beta\beta'} t_{\beta\beta'\ \alpha}{}^\gamma \lambda_\gamma\,. \end{aligned} \tag{3.16}$$

Eq. (3.14) is the starting point for our analysis of possible supergravity constraints in the tensorial superspace. In addition to (3.14) we also impose conventional constraints which express some of superfields in terms of other ones or, equivalently, fix an arbitrariness in the definition of the supervielbeins and the connection.

As this point, although well known in the context of standard supergravity [38], is important for understanding that the supergravity constraints we find are indeed the most general ones for the superspaces with *the* GL(n) *and* SL(n) *structure group*, we are going to discuss it in more detail. The reader who is not interested in technicalities may skip the next subsection 3.2.2 and pass directly to subsection 3.3.

3.2.2 On the freedom in superfield redefinitions and conventional constraints

In the case under consideration it is essential that the GL(n) structure group symmetry of the tensorial superspace allows one to make, for instance, the following redefinition of the 'spin' connection (3.1) and of the fermionic supervielbein

$$\Omega_\alpha{}^\beta \ \mapsto\ \Omega_\alpha{}^\beta + E^\gamma r_{\gamma\alpha}{}^\beta + \frac{i}{2} E^{\gamma\delta} r_{\gamma\delta\alpha}{}^\beta\,, \tag{3.17}$$

$$E^\alpha \ \mapsto\ E^\alpha - E^{\beta\gamma} S^\alpha_{\gamma\beta}\,, \tag{3.18}$$

with arbitrary superfields $r_{\gamma\alpha}{}^\beta(Z)$, $r_{\gamma\delta\alpha}{}^\beta(Z) = r_{\delta\gamma\alpha}{}^\beta(Z)$ and $S^\alpha_{\gamma\beta}(Z)$.

We now notice that the tensorial structure of components (3.14) of the bosonic torsion (3.2) is similar to that of the superfields which are used in the redefinitions (3.17) and (3.18). This allows us to simplify the torsion (3.14) by removing the $t_{\gamma\delta}{}^\beta(Z)$ superfield and also set to zero either the lowest dimensional component of the GL(n) curvature, $R_{\delta\ \gamma\alpha}{}^\beta = 0$, or alternatively to eliminate the highest dimensional component of the bosonic torsion, $t_{\gamma\gamma'\ \delta}{}^\beta(Z) = 0$. The additional conditions on the torsion and/or curvature obtained in this way are called [33] *conventional* constraints in contrast to the *essential* constraint on the torsion given by the form of the first term $(-i\, E^\alpha \wedge E^\beta)$ on the right hand side of (3.14).

Thus the two natural choices of the conventional constraints are

$$T^{\alpha\beta} = -iE^\alpha \wedge E^\beta + E^{\gamma\gamma'} \wedge E^{\delta(\alpha}\, t_{\gamma\gamma'\ \delta}{}^{\beta)}(Z)\,, \tag{3.19}$$

$$\mathcal{R}_\beta{}^\alpha = E^{\gamma\gamma'} \wedge E^\delta \mathcal{R}_{\delta\ \gamma\gamma'\beta}{}^\alpha + \frac{1}{2} E^{\gamma\gamma'} \wedge E^{\delta\delta'} \mathcal{R}_{\delta\delta'\ \gamma\gamma'\beta}{}^\alpha \tag{3.20}$$

and

$$T^{\alpha\beta} = -iE^\alpha \wedge E^\beta\,, \tag{3.21}$$

$$\mathcal{R}_\beta{}^\alpha = \frac{1}{2} E^\gamma \wedge E^\delta\ \mathcal{R}_{\gamma\delta\ \beta}{}^\alpha(Z) + E^{\gamma\gamma'} \wedge E^\delta \mathcal{R}_{\delta\ \gamma\gamma'\beta}{}^\alpha + \frac{1}{2} E^{\gamma\gamma'} \wedge E^{\delta\delta'} \mathcal{R}_{\delta\delta'\ \gamma\gamma'\beta}{}^\alpha\,. \tag{3.22}$$

One can see that the constraints (3.19), (3.20) and (3.21), (3.22) are related by the redefinition $\Omega_\alpha{}^\beta \mapsto \Omega_\alpha{}^\beta + \frac{1}{2}E^{\gamma\delta}t_{\gamma\,\delta\alpha}{}^\beta$ and $\mathcal{R}_{\gamma\,\delta\alpha}{}^\beta = -it_{\gamma\,\delta\alpha}{}^\beta$.

The consistency of the constraints (3.19), (3.20) or (3.21), (3.22) should be studied with the use of the Bianchi identities

$$\mathcal{D}T^{\alpha\beta} + E^{\alpha\gamma} \wedge \mathcal{R}_\gamma{}^\beta + E^{\beta\gamma} \wedge \mathcal{R}_\gamma{}^\alpha \equiv 0\,, \tag{3.23}$$

$$\mathcal{D}T^\alpha + E^\beta \wedge \mathcal{R}_\beta{}^\alpha \equiv 0\,, \tag{3.24}$$

$$\mathcal{D}\mathcal{R}_\beta{}^\alpha \equiv 0\,. \tag{3.25}$$

It is well known (see [38]) that although in the absence of constraints the Bianchi identities only imply that the torsion and curvature are constructed from the supervielbeins and connection, when the set of essential and conventional constraints are imposed, the Bianchi identities lead to additional restrictions on the form of the torsion and curvature, and in some cases produce dynamical equations of motion which then imply that corresponding supergravity constraints are on shell.

Also in our case the Bianchi identities impose further conditions on the form of torsion and curvature. In particular, already the study of the simplest lower dimensional component of the Bianchi identity (3.23) shows that (3.19), (3.20) (as well as (3.21), (3.22)) imply that $T_{\gamma\beta}{}^\alpha = 0$, i.e. that

$$T^\alpha = E^{\gamma\gamma'} \wedge E^\delta T_{\delta\ \gamma\gamma'}{}^\alpha + \frac{1}{2}E^{\gamma\gamma'} \wedge E^{\delta\delta'} T_{\delta\delta'\ \gamma\gamma'}{}^\alpha\,. \tag{3.26}$$

Moreover, the higher dimensional components of the Bianchi identity (3.23) imply that all the superfields in T^A and R^α_β can be expressed in terms of an antisymmetric superfield $\mathcal{R}_{\alpha\beta}$, a superfield $U_{\alpha\beta\;\gamma} = U_{\beta\alpha\;\gamma}$ and their derivatives, as we shall see in the next subsection.

3.3 The Bianchi identities and the complete set of the constraints in the tensorial superspace with the structure group GL(n)

Thus eq. (3.14) which follows from the requirement of the κ-symmetry of the tensorial superparticle and contains what is usually called *essential constraints* (in conventional superspace these are $T_{\alpha\beta}{}^a = -2i\Gamma^a_{\alpha\beta}$) is the starting point for our analysis of the supergravity constraints in tensorial superspace with the GL(n) structure group. In addition to (3.14) we also impose conventional constraints (see Subsecion 3.2.2) which express some of superfields in terms of other ones or, equivalently, fix an arbitrariness in the definition of the supervielbeins and of the connection. By imposing the conventional constraints and studying the Bianchi identities (3.23), (3.24) and (3.25) one finds the form of the torsion and curvature of the tensorial superspace. A particular choice of conventional constraints (see eqs. (3.19) and (3.20)) results in

$$T^{\alpha\beta} = -iE^\alpha \wedge E^\beta + 2E^{\gamma(\alpha} \wedge E^{\beta)\delta} R_{\gamma\delta}(Z)\,, \tag{3.27}$$

$$T^\alpha = 2E^{\alpha\beta} \wedge E^\gamma R_{\beta\gamma} + E^{\alpha\beta} \wedge E^{\gamma\delta} U_{\beta\gamma\delta}\,, \tag{3.28}$$

$$\begin{aligned} \mathcal{R}_\beta{}^\alpha = {} & iE^{\gamma\delta} \wedge E^\alpha U_{\beta\gamma\delta} - E^{\alpha\gamma} \wedge E^\delta (F_{\delta\beta\gamma} + \mathcal{D}_\delta R_{\beta\gamma}) - \\ & - E^{\alpha\gamma} \wedge E^{\delta\epsilon} (\mathcal{D}_{(\beta} U_{\gamma)\delta\epsilon} + \mathcal{D}_{\delta\epsilon} R_{\beta\gamma})\,. \end{aligned} \tag{3.29}$$

In (3.29) $R_{\gamma\delta}(Z)$ and $U_{\alpha\beta\gamma}(Z)=U_{\alpha\gamma\beta}(Z)$ are 'main' superfields[6], and

$$F_{\alpha\beta\gamma}=2iU_{(\beta\gamma)\alpha}-iU_{\alpha\beta\gamma}-2\mathcal{D}_{(\beta}R_{\gamma)\alpha}\,. \tag{3.30}$$

The main superfields are related by the equations

$$\mathcal{D}_{[\alpha}U_{\beta]\gamma\delta}\;=\;-\mathcal{D}_{\gamma\delta}R_{\alpha\beta}\;, \tag{3.31}$$
$$\mathcal{D}_{(\alpha}U_{\beta)\,\gamma\delta}\;=\;-i\mathcal{D}_{(\gamma}F_{\delta)\;\alpha\beta} \tag{3.32}$$

and

$$\mathcal{D}_{\alpha\beta}U_{\gamma\delta\sigma}-\mathcal{D}_{\delta\sigma}U_{\gamma\alpha\beta}+2U_{\gamma\alpha(\sigma}R_{\delta)\beta}+2U_{\gamma\beta(\sigma}R_{\delta)\alpha}=0\,, \tag{3.33}$$

which are the constraints on $R_{\alpha\beta}$ and $U_{\alpha\beta\gamma}$ required by the Bianchi identities (3.23) and (3.24). Due to a straightforward generalization of the Dragon theorem [32] no other independent constraints arise from the curvature Bianchi identities (3.25).

The superfields $U_{\alpha\beta\gamma}$ and $F_{\alpha\beta\gamma}$ can be alternatively expressed in terms of a totally symmetric superfield $G_{\alpha\beta\gamma}$, a derivative of $R_{\gamma\alpha}$ and a mixed symmetry superfield $H_{\alpha\beta\gamma}=H_{\alpha\gamma\beta}$ as follows

$$\begin{aligned}U_{\alpha\beta\gamma}\;&=\;G_{\alpha\beta\gamma}+\frac{2i}{3}\mathcal{D}_{(\beta}R_{\gamma)\alpha}+H_{\alpha\beta\gamma}\;,\\ -iF_{\alpha\beta\gamma}\;&=\;G_{\alpha\beta\gamma}+\frac{2i}{3}\mathcal{D}_{(\beta}R_{\gamma)\alpha}-2H_{\alpha\beta\gamma}\,.\end{aligned} \tag{3.34}$$

This decomposition is useful when we perform the reduction of GL(n) holonomy to SL(n)-holonomy, which is achieved by putting $H_{\alpha\beta\gamma}=0$ (see section 4).

Note that if we choose $R_{\alpha\beta}=-\frac{\varsigma}{2}C_{\alpha\beta}$ and $U_{(\alpha\beta\gamma)}(Z)=0$ we find that $\mathcal{R}_\alpha{}^\beta=0$, and the constraints (3.27)–(3.29) reduce to the defining relations of the Maurer-Cartan forms and of the torsion of the supergroup OSp($1|n$) in the flat basis ($\Omega_\alpha{}^\beta=0$), whose covariant derivatives form the OSp($1|n$) superalgebra (1.11), (2.10). The OSp($1|n$) Maurer-Cartan equations are

$$\begin{aligned}d\mathcal{E}^{\alpha\beta}\;&=\;-i\mathcal{E}^\alpha\wedge\mathcal{E}^\beta-\zeta\mathcal{E}^{\alpha\gamma}\wedge\mathcal{E}^{\delta\beta}C_{\gamma\delta}\;,\\ d\mathcal{E}^\alpha\;&=\;-\zeta\mathcal{E}^{\alpha\gamma}\wedge\mathcal{E}^\delta C_{\gamma\delta}\,.\end{aligned} \tag{3.35}$$

A different but equivalent set of constraints can be obtained by making the following redefinition of the connection

$$\Omega_\beta{}^\alpha\longrightarrow\Omega_\beta{}^\alpha-E^{\alpha\gamma}R_{\gamma\beta} \tag{3.36}$$

which results in the corresponding redefinition of the *vector* covariant derivative. The constraints take the form

$$T^{\alpha\beta}\;=\;-iE^\alpha\wedge E^\beta\,, \tag{3.37}$$
$$T^\alpha\;=\;E^{\alpha\beta}\wedge E^\gamma R_{\beta\gamma}+E^{\alpha\beta}\wedge E^{\gamma\delta}U_{\beta\gamma\delta}\,, \tag{3.38}$$
$$\begin{aligned}\mathcal{R}_\beta{}^\alpha\;=\;&-iE^\alpha\wedge E^\gamma R_{\beta\gamma}+iE^{\gamma\delta}\wedge E^\alpha U_{\beta\gamma\delta}-E^{\alpha\gamma}\wedge E^\delta F_{\delta\beta\gamma}-\\ &-E^{\alpha\gamma}\wedge E^{\delta\epsilon}(\mathcal{D}_{(\beta}U_{\gamma)\delta\epsilon}+R_{\beta\epsilon}R_{\gamma\delta})\,,\end{aligned} \tag{3.39}$$

[6] We hope that the reader will not confuse the curvature two-form $\mathcal{R}_\beta{}^\alpha$ with the superfield $R_{\alpha\beta}$. The notation for the latter has been chosen by analogy with $N=1$, $D=3$ supergravity where $R_{\alpha\beta}=\epsilon_{\alpha\beta}\,R$ [30]. Note also that, since we deal with the holonomy groups GL(n) and SL(n), for $n>2$ there is no metric to rise and lower the indices.

where the main superfields $U_{\alpha\beta\gamma}$ and $R_{\alpha\beta}$ satisfy the constraints

$$\mathcal{D}_{[\alpha}U_{\beta]\gamma\delta} = -\mathcal{D}_{\gamma\delta}R_{\alpha\beta}\,, \tag{3.40}$$

$$\mathcal{D}_{(\alpha}U_{\beta)\,\gamma\delta} = -i\mathcal{D}_{(\gamma}F_{\delta)\,\alpha\beta} \tag{3.41}$$

and

$$\mathcal{D}_{\alpha\beta}U_{\gamma\delta\sigma} - \mathcal{D}_{\delta\sigma}U_{\gamma\alpha\beta} + R_{\gamma(\alpha}U_{\beta)\delta\sigma} - R_{\gamma(\delta}U_{\sigma)\alpha\beta} = 0\,. \tag{3.42}$$

To conclude, eqs. (3.27), (3.28) and (3.29) describe the most general constraints on the geometry of curved tensorial superspace with the holonomy group GL(n) which are required by the tensorial superparticle with $(n-1)$ κ-symmetries. The equivalent set of constraints (3.37), (3.38) and (3.39) can be obtained by making superfield redefinitions with the use of the main superfields R and U as parameter functions.

3.4 Tensorial superspace with the holonomy group SL(n)

When $n = 2$ the constraints (3.27)–(3.29) and (3.37)–(3.39) describe conformal $N = 1$, $D = 3$ supergravity [30]. In this case the superfield $R_{\alpha\beta}$ gets reduced to the scalar density R ($R_{\alpha\beta} = \epsilon_{\alpha\beta}\, R$), and the trace part of the GL(2) connection and curvature correspond to local Weyl (scaling) symmetry. To reduce the conformal $D = 3$ supergravity to the off-shell $N = 1$, $D = 3$ Poincaré supergravity one imposes the additional tracelessness constraint on the curvature

$$\mathcal{R}_{\alpha}{}^{\alpha} = 0\,. \tag{3.43}$$

This reduces GL(2) down to SL(2) $\approx O(1,2)$ which is isomorphic to the $D = 3$ Lorentz group.

The constraint (3.43), restricting GL(n) to SL(n), can also be imposed in the general case of $n \geq 2$. Then the main superfields reduce to

$$-\,iF_{\alpha\beta\gamma} = U_{\alpha\beta\gamma} = G_{\alpha\beta\gamma} + \frac{2i}{3}\mathcal{D}_{(\beta}R_{\gamma)\alpha}\,, \tag{3.44}$$

where $G_{\alpha\beta\gamma}$ is totally symmetric. In view of (3.34) we observe that the condition of SL(n) holonomy amounts to putting to zero the tensor $H_{\alpha\beta\gamma}$.

The superfields U, G and R satisfy the following differential relations

$$\mathcal{D}_{(\alpha}U_{\beta)\gamma\delta} = \mathcal{D}_{(\gamma}U_{\delta)\alpha\beta}\,, \tag{3.45}$$

(which is a consequence of (3.32) and (3.44)), and

$$\mathcal{D}_{[\alpha}G_{\beta]\gamma\delta} = -\mathcal{D}_{\gamma\delta}R_{\alpha\beta} - \frac{i}{3}\left(\mathcal{D}_{\alpha}\mathcal{D}_{(\gamma}R_{\delta)\beta} - \mathcal{D}_{\beta}\mathcal{D}_{(\gamma}R_{\delta)\alpha}\right). \tag{3.46}$$

Since $G_{\beta\gamma\delta}$ is totally symmetric, from eq. (3.46) we can get

$$2\mathcal{D}_{[\alpha}G_{\beta]\gamma\delta} = -\mathcal{D}_{(\gamma\delta}R_{\alpha)\beta} + \mathcal{D}_{(\gamma\delta}R_{\beta)\alpha}\,. \tag{3.47}$$

To derive (3.47) we first symmetrize the left and the right hand side of (3.46) in $(\gamma\delta\alpha)$ and then sum up the result with the symmetrization of (3.46) in $(\gamma\delta\beta)$.

Comparing (3.46) with (3.47) we find a condition which must be satisfied by $R_{\alpha\beta}$ and which will appear in section 4 as part of the integrability of a scalar superfield equation in an external tensorial supergravity background. This condition can also be obtained by antisymmetrizing the indices $[\alpha\beta\gamma]$ in (3.46) which gives

$$\mathcal{D}_{[\alpha}\mathcal{D}_{\beta}R_{\gamma]\delta} + \mathcal{D}_{\delta}\mathcal{D}_{[\alpha}R_{\beta\gamma]} = 5i\mathcal{D}_{\delta[\alpha}R_{\beta\gamma]}\,. \tag{3.48}$$

Then symmetrizing eq. (3.48) with respect to $(\gamma\delta)$ and regrouping indices we get

$$\mathcal{D}_{\gamma}\mathcal{D}_{[\alpha}R_{\beta]\delta} + \mathcal{D}_{\delta}\mathcal{D}_{[\alpha}R_{\beta]\gamma} = 2i\mathcal{D}_{\gamma\delta}R_{\alpha\beta} + 3i\mathcal{D}_{\gamma[\alpha}R_{\beta]\delta} + 3i\mathcal{D}_{\delta[\alpha}R_{\beta]\gamma}\,. \tag{3.49}$$

We shall encounter this last form of the condition on $R_{\alpha\beta}$ in section 4 analyzing the consistency of the propagation of a scalar field in a non-linear tensorial supergravity background.

Let us also note that using the anticommutation relation $\{\mathcal{D}_{\alpha}, \mathcal{D}_{\alpha}\} = 2i\mathcal{D}_{\alpha\beta}$, from (3.49) one finds that the last two terms in the right hand side of (3.46) can be rewritten in the form

$$\mathcal{D}_{\alpha}\mathcal{D}_{(\gamma}R_{\delta)\beta} - \mathcal{D}_{\beta}\mathcal{D}_{(\gamma}R_{\delta)\alpha} = 2i\mathcal{D}_{\gamma\delta}R_{\alpha\beta} + i\mathcal{D}_{\gamma[\alpha}R_{\beta]\delta} + i\mathcal{D}_{\delta[\alpha}R_{\beta]\gamma} \tag{3.50}$$

which upon the substitution into (3.46) gives (3.47). This can be regarded as a check or as an alternative derivation of eq. (3.47).

Eq. (3.48) is identically satisfied in the case of $N = 1$, $D = 3$ supergravity (in which case $\alpha, \beta, \gamma = 1, 2$, and hence the antisymmetrization of three indices gives zero), but it is nontrivial for the tensorial superspaces with $n > 2$.

Using (3.44), from eq. (3.45) one derives another consequence of the constraint (3.43)

$$\mathcal{D}_{[\alpha}G_{\beta]\gamma\delta} + \mathcal{D}_{[\gamma}G_{\delta]\alpha\beta} = -\frac{2}{3}\left(\mathcal{D}_{\delta(\alpha}R_{\gamma)\beta} + \mathcal{D}_{\beta(\alpha}R_{\gamma)\delta}\right). \tag{3.51}$$

In view of (3.47) the equation (3.51) (and hence (3.45)) is identically satisfied and therefore does not put further restrictions on the form of $R_{\alpha\beta}$ and $G_{\alpha\beta\gamma}$.

To conclude, when the holonomy group is restricted to SL(n) by (3.43), the constraints (3.27) and (3.28) remain the same

$$\begin{aligned} T^{\alpha\beta} &= -iE^{\alpha}\wedge E^{\beta} + 2E^{\gamma(\alpha}\wedge E^{\beta)\delta}R_{\gamma\delta}\,, \\ T^{\alpha} &= 2E^{\alpha\beta}\wedge E^{\gamma}R_{\beta\gamma} + E^{\alpha\beta}\wedge E^{\gamma\delta}U_{\beta\gamma\delta} \end{aligned} \tag{3.52}$$

while (3.29) reduce to

$$\begin{aligned} \mathcal{R}_{\beta}{}^{\alpha} &= iE^{\gamma\delta}\wedge E^{\alpha}U_{\beta\gamma\delta} - E^{\alpha\gamma}\wedge E^{\delta}(iU_{\delta\beta\gamma} + \mathcal{D}_{\delta}R_{\beta\gamma}) - \\ &- E^{\alpha\gamma}\wedge E^{\delta\epsilon}(\mathcal{D}_{(\beta}U_{\gamma)\delta\epsilon} + \mathcal{D}_{\delta\epsilon}R_{\beta\gamma})\,. \end{aligned} \tag{3.53}$$

The superfield $U_{\alpha\beta\gamma}$ is expressed through the totally symmetric superfield $G_{\alpha\beta\gamma}$ and a derivative of the superfield $R_{\alpha\beta}$ by the equation (3.44), the main superfields $G_{\alpha\beta\gamma}$ and $R_{\alpha\beta}$ being related to and constrained by eqs. (3.46) and (3.33).

We should note that further reduction of the SL(n) holonomy group down to its subgroup Sp(n) imposes in the case of $n > 2$ additional restrictions on $R_{\alpha\beta}$ and $G_{\alpha\beta\gamma}$ which trivialize the tensorial supergravity down to either flat tensorial superspace or the supergroup manifold OSp($1|n$).

In the case of the $N = 1$, $D = 3$ supergravity (where $n = 2$) SL(2) is isomorphic to Sp(2), the constraints (3.52)–(3.53) are off the mass shell and the trivialization does not occur. The supergravity equations of motion are obtained by putting

$$R_{\alpha\beta} = 0, \qquad G_{\alpha\beta\gamma} = 0\,, \tag{3.54}$$

or in the case of AdS

$$R_{\alpha\beta} = -\frac{\varsigma}{2} C_{\alpha\beta}\,, \qquad G_{\alpha\beta\gamma} = 0\,. \tag{3.55}$$

These equations imply that pure $N = 1$, $D = 3$ supergravity is non-dynamical, since its torsion and curvature vanish [30].

Also in the case of $n > 2$ the equations (3.54) or (3.55) single out, respectively, the flat or $\mathrm{OSp}(1|n)$ vacuum solution of the tensorial supergravity constraints.

4. The scalar superfield equation in a tensorial supergravity background

In the previous sections we have derived the constraints of tensorial supergravity from the requirement of the κ-symmetry of the "preonic" superparticle. The supergravity constraints can also be obtained (see [33, 38] for the ordinary case) by requiring that in curved superspace there exist (super)field representations of (generalized) supersymmetry similar to those in flat superspace. In our case of flat tensorial superspace and of $\mathrm{OSp}(1|n)$ the only known representation is described by the scalar superfield obeying the dynamical equations (2.3) and (2.11), respectively. So it is natural to consider a curved superspace generalization of these equations and to analyze which restrictions on superspace geometry are imposed by its integrability. Instead of starting again from the most general structure of tensorial supergeometry, in this section we shall consider a possible generalization of eqs. (2.3) and (2.11) in a curved superspace already subject to the supergravity constraints (3.27), (3.28) and (3.29). Interestingly enough, the integrability of the scalar superfield equation will require the curved superspace holonomy to be $\mathrm{SL}(n)$ and not $\mathrm{GL}(n)$.

A natural generalization of the free superfield equations (2.3) and (2.11) is

$$\mathcal{D}_{[\alpha}\mathcal{D}_{\beta]}\,\Phi = \frac{i}{2}\,R_{\alpha\beta}\,\Phi\,. \tag{4.1}$$

One gets eqs. (2.3) and (2.11) from (4.1) by putting $U_{\alpha\beta\gamma} = 0$ and $R_{\alpha\beta} = 0$ or $R_{\alpha\beta} = -\frac{\varsigma}{2}C_{\alpha\beta}$, respectively[7]. A more general form of the scalar superfield equation is discussed in appendix B.

Let us now study the integrability of the equations (4.1) in the case of supergravity with the holonomy group $\mathrm{GL}(n)$ subject to the constraints (3.27)–(3.29). To this end we need the following covariant derivative commutation relations (where W_δ is an arbitrary superfield)

$$\{\mathcal{D}_\alpha\,,\,\mathcal{D}_\beta\} = 2i\mathcal{D}_{\alpha\beta}\,, \tag{4.2}$$

$$[\mathcal{D}_{\alpha\beta}\,,\,\mathcal{D}_\gamma]W_\delta = -2R_{\gamma(\alpha}\mathcal{D}_{\beta)}W_\delta - iU_{\delta\alpha\beta}W_\gamma + F_{\gamma\delta(\alpha}W_{\beta)} + \mathcal{D}_\gamma R_{\delta(\alpha}W_{\beta)}\,, \tag{4.3}$$

[7]Note that eq. (4.1) resembles a conformally invariant scalar field equation in a $D = 4$ gravitational background $g^{mn}\mathcal{D}_m\partial_n\,b(x) = \frac{1}{6}\,R(x)\,b(x)$, where $R(x)$ is the curvature scalar.

and

$$[\mathcal{D}_{\alpha\beta}\,,\,\mathcal{D}_{\gamma\delta}]W_\epsilon = -4R_{(\gamma|(\alpha}\mathcal{D}_{\beta)|\delta)}W_\epsilon + U_{(\alpha|\,\gamma\delta}\mathcal{D}_{|\beta)}W_\epsilon - U_{(\gamma|\,\alpha\beta}\mathcal{D}_{\delta)}W_\epsilon + \\ + \mathcal{D}_{\gamma\delta}R_{\epsilon(\alpha}\,W_{\beta)} - \mathcal{D}_{\alpha\beta}R_{\epsilon(\gamma}\,W_{\delta)} + \frac{1}{2}\mathcal{D}_\epsilon U_{(\alpha|\,\gamma\delta}\,W_{|\beta)} - \\ - \frac{1}{2}\mathcal{D}_\epsilon U_{(\gamma|\,\alpha\beta}\,W_{\delta)} - \frac{1}{2}W_{(\beta}\mathcal{D}_{\alpha)}U_{\epsilon\,\gamma\delta} + \frac{1}{2}W_{(\delta}\mathcal{D}_{\gamma)}U_{\epsilon\,\alpha\beta}\,, \tag{4.4}$$

where it is implied that the indices $(\alpha\beta)$ as well as $(\gamma\delta)$ are symmetrized with the unit weight ($A_{\alpha\beta} = A_{(\alpha\beta)} + A_{[\alpha\beta]}$). Acting on (4.1) with $\mathcal{D}_\gamma$ and using (4.2) and (4.3) we get the non-linear counterpart of the fermionic equations (1.7) and (2.13)

$$\mathcal{D}_{\alpha[\beta}\mathcal{D}_{\gamma]}\Phi = \frac{1}{2}\left(R_{\alpha[\gamma}\mathcal{D}_{\beta]}\Phi + R_{\beta\gamma}\mathcal{D}_\alpha\Phi\right) - \frac{\Phi}{6}\,\mathcal{D}_{[\beta}R_{\gamma]\alpha} + \frac{\Phi}{6}\,\mathcal{D}_\alpha R_{\beta\gamma}\,. \tag{4.5}$$

Acting on (4.5) with $\mathcal{D}_\delta$ and using the commutation relations (4.2) and (4.3) we get the non-linear counterpart of the bosonic equations (1.6) and (2.12)

$$\mathcal{D}_{\alpha[\beta}\mathcal{D}_{\gamma]\delta}\Phi = \frac{1}{2}\left(R_{\beta\gamma}\mathcal{D}_{\alpha\delta} - R_{\alpha[\beta}\mathcal{D}_{\gamma]\delta} - R_{\delta[\beta}\mathcal{D}_{\gamma]\alpha}\right)\Phi + \frac{1}{4}\left(R_{\alpha\delta}R_{\beta\gamma} - R_{\alpha[\beta}R_{\gamma]\delta}\right)\Phi + \\ + \frac{i}{6}\mathcal{D}_\alpha R_{\beta\gamma}\mathcal{D}_\delta\Phi - \frac{i}{6}\mathcal{D}_{[\beta}R_{\gamma]\alpha}\mathcal{D}_\delta\Phi + \frac{i}{2}\mathcal{D}_\alpha R_{\delta[\beta}\mathcal{D}_{\gamma]}\Phi - \frac{i}{2}\mathcal{D}_{[\beta}\Phi\,\mathcal{D}_{\gamma]}R_{\alpha\delta} + \\ + \frac{\Phi}{6}(3\mathcal{D}_{\alpha[\beta}\,R_{\gamma]\delta} - i\mathcal{D}_\delta\mathcal{D}_\alpha R_{\beta\gamma} + i\mathcal{D}_\delta\mathcal{D}_{[\beta}R_{\gamma]\alpha}) + \\ + U_{[\beta\gamma]\alpha}\mathcal{D}_\delta\Phi + \frac{1}{2}U_{(\gamma\alpha)\delta}\mathcal{D}_\beta\Phi - \frac{1}{2}U_{(\beta\alpha)\delta}\mathcal{D}_\gamma\Phi + \frac{1}{2}U_{\delta\alpha[\beta}\mathcal{D}_{\gamma]}\Phi\,. \tag{4.6}$$

On the other hand, as a consequence of the constraints (3.27) and (3.28) the 'antisymmetrized' commutator of the bosonic covariant derivatives (4.4) acting on a scalar superfield has the following form

$$\frac{1}{2}[\mathcal{D}_{\alpha[\beta},\mathcal{D}_{\gamma]\delta}]\,\Phi = \frac{1}{2}(\mathcal{D}_{\alpha[\beta}\mathcal{D}_{\gamma]\delta} + \mathcal{D}_{\delta[\beta}\mathcal{D}_{\gamma]\alpha})\Phi \tag{4.7}$$

$$= -\frac{1}{2}(R_{\alpha[\beta}\mathcal{D}_{\gamma]\delta}\Phi + R_{\delta[\beta}\mathcal{D}_{\gamma]\alpha}\,\Phi - R_{\beta\gamma}\mathcal{D}_{\alpha\delta}\,\Phi) + \frac{1}{2}\left(U_{[\beta\gamma](\alpha}\mathcal{D}_{\delta)}\Phi - U_{(\alpha\delta)[\beta}\mathcal{D}_{\gamma]}\Phi\right).$$

From (4.7) it follows that for the equation (4.1) to be consistent, the right hand side of the equation (4.6) symmetrized in α and δ must coincide with the right hand side of (4.7). This results in the equation

$$0 = U_{[\beta\gamma](\alpha}\,\mathcal{D}_{\delta)}\Phi - \frac{i}{3}\,\mathcal{D}_{(\alpha}\Phi\mathcal{D}_{\delta)}R_{\beta\gamma} - \frac{i}{3}\mathcal{D}_{[\beta}R_{\gamma](\alpha}\,\mathcal{D}_{\delta)}\Phi + \\ + U_{(\alpha\,\delta)[\beta}\,\mathcal{D}_{\gamma]}\Phi + \mathcal{D}_{[\gamma}\Phi\,U_{\beta]\,\alpha\delta} + \mathcal{D}_{(\alpha}R_{\delta)[\beta}\,\mathcal{D}_{\gamma]}\Phi + \\ + \frac{\Phi}{12}\left(\mathcal{D}_{\alpha\delta}R_{\beta\gamma} + 3\mathcal{D}_{\alpha[\beta}R_{\gamma]\delta} + i\mathcal{D}_\alpha\mathcal{D}_{[\beta}R_{\gamma]\delta} + \alpha\leftrightarrow\delta\right). \tag{4.8}$$

The above equation is identically satisfied in the case of the tensorial supergravity with the holonomy group SL(n) described in subsection 3.4. Indeed, the first and the second line in (4.8) vanish in virtue of eq. (3.44), while the last line coincides with the left hand side of eq. (3.49).

Thus, the scalar superfield can consistently propagate in any tensorial supergravity background with SL(n) holonomy. Peculiarities of the coupling of a scalar superfield to $N = 1$, $D = 3$ supergravity are briefly discussed in appendix C.

5. Generalized Weyl invariance of the tensorial supergravity constraints and conformally related supermanifolds

Let us now proceed with studying the properties of the tensorial supergravity constraints and looking for their general solution in terms of an unconstrained superfield. To this end consider the following transformations of the supervielbeins and superconnection of a tensorial superspace

$$\begin{aligned} E'^{\alpha\beta} &= E^{\alpha\beta}\,, \\ E'^{\alpha} &= E^{\alpha} + E^{\alpha\beta}W_\beta \\ \Omega'_\beta{}^{\alpha} &= \Omega_\beta{}^\alpha - iE^\alpha W_\beta - E^{\alpha\gamma}(\mathcal{D}_\gamma W_\beta + iW_\gamma W_\beta)\,, \end{aligned} \tag{5.1}$$

where W_α is an arbitrary spinor superfield. Then, as one can check, the form of the supergravity torsion and curvature (3.27)–(3.29) remain intact when the transformed $R'_{\alpha\beta}$ and $U'_{\alpha\beta\gamma}$ are defined as

$$\begin{aligned} R'_{\alpha\beta} &= R_{\alpha\beta} - \mathcal{D}_{[\alpha}W_{\beta]} - \frac{i}{2}W_\alpha W_\beta\,, \\ U'_{\alpha\beta\gamma} &= U_{\alpha\beta\gamma} + \mathcal{D}_{\beta\gamma}W_\alpha - W_{(\gamma}\mathcal{D}_{\beta)}W_\alpha\,. \end{aligned} \tag{5.2}$$

As a result, the main superfields $R'_{\alpha\beta}$ and $U'_{\alpha\beta\gamma}$ satisfy the constraints (3.31)–(3.33) provided that $R_{\alpha\beta}$ and $U_{\alpha\beta\gamma}$ solve them and vice versa.

Thus the solutions of the supergravity constraints (3.27)–(3.29) form classes of equivalence whose members are related by the transformations (5.1)–(5.2). These can be regarded as a kind of generalized super-Weyl transformations which reduce to proper Weyl transformations when $W_\alpha = -i\mathcal{D}_\alpha W(Z)$ with $W(Z)$ being a scalar superfield (see e.g. [34, 35]).

In particular, when $R'_{\alpha\beta} = 0 = U'_{\alpha\beta\gamma}$ correspond to the flat superspace, eqs. (5.2) describe a class of conformally flat tensorial superspaces whose holonomy group is SL(n) if $W_\alpha = -i\mathcal{D}_\alpha W(Z)$

$$\begin{aligned} R_{\alpha\beta} &= \mathcal{D}_{[\alpha}W_{\beta]} + \frac{i}{2}W_\alpha W_\beta\,, \\ U_{\alpha\beta\gamma} &= -\mathcal{D}_{\beta\gamma}W_\alpha + W_{(\gamma}\mathcal{D}_{\beta)}W_\alpha\,. \end{aligned} \tag{5.3}$$

To see this let us calculate the trace of the curvature (3.29) with $R_{\alpha\beta}$ and $U_{\alpha\beta\gamma}$ given by eqs. (5.3). The trace takes the form

$$\mathcal{R}_{\alpha\beta\gamma,\,\delta}{}^{\delta} = -i\mathcal{D}_{\beta\gamma}W_\alpha + \mathcal{D}_\alpha\mathcal{D}_{(\beta}W_{\gamma)} + 2iW_{(\beta}R_{\gamma)\alpha}\,. \tag{5.4}$$

For a generic W_α the trace of the curvature is non-zero, however it identically vanishes when $W_\alpha = -i\mathcal{D}_\alpha W$. Indeed in this case, in virtue of the commutation relations

$$\{\mathcal{D}_\alpha\,,\,\mathcal{D}_\beta\} = 2i\mathcal{D}_{\alpha\beta}\,, \tag{5.5}$$

$$[\mathcal{D}_{\alpha\beta}\,,\,\mathcal{D}_\gamma]W = -2R_{\gamma(\alpha}\mathcal{D}_{\beta)}W\,, \tag{5.6}$$

we get

$$\mathcal{R}_{\alpha\beta\gamma,\,\delta}{}^{\delta} = -[\mathcal{D}_{\beta\gamma},\mathcal{D}_\alpha]\,W - 2\,R_{\alpha(\beta}\mathcal{D}_{\gamma)}\,W \equiv 0\,. \tag{5.7}$$

A simpler way to arrive at the same conclusion is to calculate the trace of the connection in (5.1), $\Omega'^{\ \alpha}_{\alpha} = \Omega_\alpha{}^\alpha - iE^\alpha W_\alpha - E^{\alpha\beta}\mathcal{D}_\alpha W_\beta$. With $W_\alpha = -i\mathcal{D}_\alpha W$ and $\Omega'^{\ \alpha}_{\alpha} = 0$ this gives $\Omega_\alpha{}^\alpha = dW$ which implies $\mathcal{R}_\alpha{}^\alpha = 0$.

In conventional superfield theories, a detailed analysis of which superspaces among supermanifolds containing $AdS_d \times S^m$ are superconformally flat has been carried out in [35]. For instance, it was demonstrated that the $N=1$ supersymmetric AdS_3 isomorphic to OSp(1|2) is superconformally flat. This is also a particular case of the tensorial superspace under consideration when $n=2$, $R_{\alpha\beta} = -\frac{\varsigma}{2}\epsilon_{\alpha\beta}$ and $U_{\alpha\beta\gamma} = 0$.

In [24, 25] it has been found that OSp(1|n) are so called GL(n) flat supermanifolds, i.e. their bosonic supervielbeins $E^{\alpha\beta}$ are obtained from the flat ones by a transformation with a certain GL(n) matrix, while the fermionic supervielbeins E^α have a more sophisticated form than that of (5.1). For $n=2$ the two properties, superconformal flatness and GL-flatness, are equivalent since GL(2) $\sim$ SL(2) $\times R$ and SL(2) $\sim Sp(2)$ is the holonomy group of OSp(1|2). As we have already discussed a supergroup manifold OSp(1|n) with $n>2$ has the holonomy group Sp(n) which is smaller than SL(n), therefore in the generic case the properties of superconformal flatness and of GL-flatness (which, in the way it works, preserves Sp(n)-holonomy) are not equivalent and hence do not imply each other.

Indeed, the supergroup manifold OSp(1|n) with $n>2$ is not superconformally flat. To show this let us recall again that the main superfields $R_{\alpha\beta}$ and $G_{\alpha\beta\gamma}$ of OSp(1|n) satisfy the equations (3.55) and $U_{\alpha\beta\gamma} = 0$ which imply that

$$\mathcal{D}_{(\alpha}\, C_{\beta)\gamma} = D_{(\alpha}\, C_{\beta)\gamma} - \Omega_{(\alpha\,\beta)}{}^{\delta}\, C_{\gamma\delta} + C_{\delta(\alpha}\, \Omega_{\beta)\,\gamma}{}^{\delta} = 0 \qquad \Rightarrow \qquad \Omega_{(\alpha\,[\beta)}{}^{\delta}\, C_{\gamma]\delta} = 0\,. \tag{5.8}$$

Substituting into (5.8) the superconformally flat form of the connection (see eq. (6.2) of section 5) we arrive at the condition

$$(n-2)(n+1)\, D_\alpha\, W = 0\,, \tag{5.9}$$

from which it follows that the Weyl scalar superfield $W(Z)$ does not reduce to the constant only when $n=2$. When $n>2$, $W = const$ and thus (3.55) are consistent with (5.3) if only $\varsigma = 0$. Hence, OSp(1|n) with $n>2$ is not superconformally flat.

As so, in the case, when $R'_{\alpha\beta} = -\frac{\varsigma}{2}C_{\alpha\beta}$ and $U'_{\alpha\beta\gamma} = 0$ which are that of the supermanifold OSp(1|n), the generalized Weyl transformations produce a class of tensorial superspaces which are not superconformally flat but are conformally related to OSp(1|n) with $R_{\alpha\beta}$ and $U_{\alpha\beta\gamma}$ having the following form

$$R_{\alpha\beta} = -\frac{\varsigma}{2}C_{\alpha\beta} + \mathcal{D}_{[\alpha}\, W_{\beta]} + \frac{i}{2}\, W_\alpha W_\beta\,, \tag{5.10}$$

$$U_{\alpha\beta\gamma} = -\mathcal{D}_{\beta\gamma}\, W_\alpha + W_{(\gamma}\, \mathcal{D}_{\beta)}\, W_\alpha\,. \tag{5.11}$$

We should note that though the supermanifold OSp(1|n) we started with has the holonomy group Sp(n) with respect to which $C_{\alpha\beta}$ is covariantly constant, the resulted supermanifolds described by (5.10) and (5.11) have (in general) the holonomy group GL(n) which reduces to SL(n) when $W_\alpha = -i\mathcal{D}_\alpha\, W(Z)$. With respect to the GL(n)- or SL(n)-covariant derivative $C_{\alpha\beta}$ is not covariantly constant. Hence, the equation (5.10) should be considered as valid in some gauge which reduces GL(n) or SL(n) down to Sp(n). The GL(n) covariant expression for $R_{\alpha\beta}$ is

$$R_{\alpha\beta} = -\frac{\varsigma}{2}\mathcal{X}_{\alpha\beta}(Z) + \mathcal{D}_{[\alpha}\, W_{\beta]} + \frac{i}{2}\, W_\alpha W_\beta\,, \tag{5.12}$$

where $\mathcal{X}_{\alpha\beta}(Z)$ is now an antisymmetric tensor superfield with $\det \mathcal{X}_{\alpha\beta} \neq 0$. Using a local $\mathrm{GL}(n)$ transformation $\mathcal{X}'_{\alpha\beta}(Z) = G_\alpha{}^{\alpha'}(Z)\, G_\beta{}^{\beta'}(Z)\, \mathcal{X}_{\alpha'\beta'}(Z)$ it is always possible to put $\mathcal{X}'_{\alpha\beta}(Z) = C_{\alpha\beta}$ and to reduce (5.12) to (5.10).

6. Decoupling of higher spin field dynamics from superconformal geometry

Since the higher spin fields (at least at the linearized level) are described by the single scalar superfield it is natural to assume that in order to switch on non-trivial higher spin interactions the geometry of tensorial supergravity coupled to the scalar superfield is itself expressed in terms of this single scalar superfield (see [5] for a somewhat similar assumption in the framework of the unfolded higher spin formulation). If we restrict ourselves to the class of conformally flat manifolds or to the class of manifolds conformally related to $\mathrm{OSp}(1|n)$ discussed in the previous section and assume that the geometry is expressed in term of a single physical scalar superfield Φ we find that the geometry reduces to flat superspace (or to $\mathrm{OSp}(1|n)$ superspace) and the superfield Φ (in a sense) decouples from the geometry. The reason is in the generalized super-Weyl invariance of both the supergravity constraints and the scalar superfield equation (4.1).

Let us first discuss the conformally flat case and then the $\mathrm{OSp}(1|n)$ related one.

Using the generic expressions of section 4 the supervielbeins and the $\mathrm{SL}(n)$ connection of a conformally flat superspace can be presented in the following conventional form (cf. e.g. [34, 35])

$$E^{\alpha\beta} = e^{\frac{2W(Z)}{n}}\, E_0^{\alpha'\beta'}\, L_{\alpha'}^{\alpha}(Z)\, L_{\beta'}^{\beta}(Z)\,, \quad E^{\alpha} = e^{\frac{W(Z)}{n}}\, (E_0^{\alpha'} - iE_0^{\alpha'\beta'}\, D_{\beta'}\, W)\, L_{\alpha'}^{\alpha}(Z), \tag{6.1}$$

$$\Omega_\beta{}^\alpha = \Omega_0{}_\beta{}^\alpha + \frac{1}{n} dW\, \delta_\beta{}^\alpha - L^{-1}{}_\beta{}^{\beta'} \left[E_0^{\alpha'}\, D_{\beta'}W + E_0^{\alpha'\gamma}(D_{\gamma\beta'}W + \frac{i}{2}D_\gamma W\, D_{\beta'}W)\right] L_{\alpha'}{}^{\alpha}\,,$$
$$\Omega_\alpha{}^\alpha \equiv 0\,, \tag{6.2}$$

where $L_\beta{}^\alpha(Z)$ is a matrix of local $\mathrm{SL}(n)$ transformations which together with $e^{\frac{W(Z)}{n}}$ form a $\mathrm{GL}(n)$ matrix $G_\beta{}^\alpha = e^{\frac{W(Z)}{n}} L_\beta{}^\alpha$. The supervielbeins $E_0^{\alpha\beta}$, E_0^α and the connection $\Omega_0{}_\beta{}^\alpha$ satisfy the constraints of a flat superspace

$$T_0^{\alpha\beta} = -iE_0^\alpha \wedge E_0^\beta, \qquad T_0^\alpha = 0 = \mathcal{R}_0{}_\beta{}^\alpha \tag{6.3}$$

and $D_{\alpha\beta}$ and D_α are corresponding covariant derivatives.

In particular, in the 'flat' basis

$$E_0^{\alpha\beta} = dX^{\alpha\beta} - id\theta^{(\alpha}\,\theta^{\beta)}, \qquad E_0^\alpha = d\theta^\alpha\,, \quad \Omega_0{}_\beta{}^\alpha = 0\,, \qquad L_\beta{}^\alpha = \delta_\beta{}^\alpha\,, \tag{6.4}$$

$$D_{\alpha\beta} = \frac{\partial}{\partial X^{\alpha\beta}} \equiv \partial_{\alpha\beta}\,, \qquad D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\theta^\beta\,\partial_{\beta\alpha} \tag{6.5}$$

and

$$\mathcal{D}_{\alpha\beta} = e^{-\frac{2W}{n}}(D_{\alpha\beta} - iD_{(\alpha}W\,D_{\beta)}) + \Omega_{\alpha\beta} - ie^{-\frac{W}{n}}D_{(\alpha}W\,\Omega_{\beta)}\,, \quad \mathcal{D}_\alpha = e^{-\frac{W}{n}}D_\alpha + \Omega_\alpha\,, \tag{6.6}$$

Using the constraint relations (3.37)–(3.39) and (3.43) one finds that in the 'flat' basis the main superfields $R_{\alpha\beta}$ and $U_{\beta\gamma\delta}$ have the form

$$R_{\alpha\beta} = i\,e^{-\frac{2W}{n}}\left[D_{[\alpha}D_{\beta]}\,W + \frac{1}{2}\,D_\alpha W\,D_\beta W\right], \tag{6.7}$$

$$U_{\beta\,\gamma\delta} = e^{-\frac{3W}{n}}\left[-i\partial_{\gamma\delta}D_\beta W + D_{(\gamma}W\,D_{\delta)}D_\beta W\right], \tag{6.8}$$

or in the basis of the 'curved' covariant derivatives (6.6)

$$\begin{aligned} R_{\alpha\beta} &= i\mathcal{D}_{[\alpha}\mathcal{D}_{\beta]}\,W - \frac{i}{2}\,\mathcal{D}_\alpha W\,\mathcal{D}_\beta W \\ &= -\mathcal{D}_{[\alpha}W_{\beta]} + \frac{i}{2}\,W_\alpha\,W_\beta\,, \end{aligned} \tag{6.9}$$

$$\begin{aligned} U_{\beta\,\gamma\delta} &= i\mathcal{D}_{\gamma\delta}\mathcal{D}_\beta\,W - \mathcal{D}_{(\gamma}\,W\,\mathcal{D}_{\delta)}\mathcal{D}_\beta\,W \\ &= -\mathcal{D}_{\gamma\delta}W_\beta + W_{(\gamma}\mathcal{D}_{\delta)}W_\beta\,. \end{aligned} \tag{6.10}$$

where we have introduced $W_\alpha \equiv -i\mathcal{D}_\alpha W$ to compare these expressions with those of section 4.

In view of a generic reasoning behind the constraint conserving transformations of the supervielbeins and superconnection given in section 4 one can directly check that the main superfields (6.7)–(6.10) identically satisfy the constraints (3.37)–(3.43) of tensorial supergravity with SL(n) holonomy, which can be checked directly.

Now our assumption that the geometry depends only on the scalar superfield Φ implies that W becomes a scalar function of Φ, $W = W(\Phi)$, and using this (physical) scalar superfield $W(\Phi)$, we are allowed to perform the Weyl transformation (5.1) and get for the transformed superfields $R'_{\alpha\beta} = 0 = U'_{\alpha\beta\gamma}$, i.e. flat superspace[8]. Thus the superfield Φ decouples from supergravity, and the most general form of the scalar superfield equation which one may construct in such a case is

$$D_{[\alpha}D_{\beta]}\,\Phi = \mathcal{X}_{\alpha\beta}(\Phi)\,, \tag{6.11}$$

where $\mathcal{X}_{\alpha\beta}(\Phi)$ is an antisymmetric tensor which depends on Φ and its derivatives. $\mathcal{X}_{\alpha\beta}(\Phi)$ must satisfy an integrability condition (see (B.4) of the appendix B, where one should put $\mathcal{D}_\alpha = D_\alpha$, $\mathcal{D}_{\alpha\beta} = \partial_{\alpha\beta}$ and $R_{\alpha\beta} = 0$).

[8]Note that if we *formally* put to zero only $R_{\alpha\beta}$ while keeping $U_{\alpha\beta\gamma}$ in the form (6.10) we get

$$R_{\alpha\beta} = 0 \qquad \Rightarrow \qquad D_{[\alpha}D_{\beta]}\,W + \frac{1}{2}\,D_\alpha W\,D_\beta W = 0\,.$$

This equation reduces to the free scalar superfield equation (2.3) upon the field redefinition $W = 2\ln\Phi$, or better $W = 2\ln(\Phi + a)$ with an arbitrary constant $a > 0$. So one might think that at least free higher spin dynamics is intrinsically encoded in superconformally flat tensorial geometry, but this is not the case since using the super-Weyl transformations with a parameter satisfying the free scalar superfield equation one can put $U_{\alpha\beta\gamma} = 0$ and arrive in flat superspace with no dynamics.

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If, for example, we choose $\mathcal{X}_{\alpha\beta}(\Phi) = -f'(\Phi) D_\alpha \Phi D_\beta \Phi$, where $f(\Phi)$ is an arbitrary function and $f'(\Phi) = \frac{df}{d\Phi}$, the eq. (6.11) takes the form

$$D_{[\alpha} D_{\beta]}\, \Phi + f'(\Phi) D_\alpha \Phi D_\beta \Phi = 0\,,$$

which upon the field redefinition $\tilde{\Phi} = const \cdot \int d\Phi\, e^{f(\Phi)}$ (i.e. $\frac{d\tilde{\Phi}}{d\Phi} = const \cdot e^{f(\Phi)}$) reduces to the free scalar superfield equation (2.3).

If there exists a more general $\mathcal{X}_{\alpha\beta}(\Phi)$ satisfying the consistency condition (see eq. (B.4) of appendix B), the equation (6.11) would describe a non-linear dynamics of a self-interacting scalar superfield $\Phi(Z)$. Since, as we have explained in the Introduction, $\Phi(Z)$ contains only the linearized field strengths of the higher spin fields and not their potentials, such a non-linear dynamics of higher spin fields would not include minimal coupling terms which require potentials or connections. It would contain only terms constructed of higher orders of the higher spin field strengths. As a result, the non-linear model obtained in this way would be analogous in a certain sense to the abelian Dirac-Born-Infeld theory.

Let us now discuss the case of the tensorial manifolds conformally related to OSp$(1|n)$. The consideration follows the same lines as in the case of the conformally flat manifolds and the result is that in terms of the OSp$(1|n)$ covariant derivatives the main tensor fields describing their geometry have the following form

$$R_{\alpha\beta} = i\, e^{-\frac{2W}{n}} \left[i\frac{\varsigma}{2} C_{\alpha\beta} + \nabla_{[\alpha} \nabla_{\beta]}\, W + \frac{1}{2}\, \nabla_\alpha W\, \nabla_\beta W \right], \tag{6.12}$$

$$U_{\beta\gamma\delta} = e^{-\frac{3W}{n}} \left[-i \nabla_{\gamma\delta} \nabla_\beta W + \nabla_{(\gamma} W\, \nabla_{\delta)} \nabla_\beta W \right]. \tag{6.13}$$

With such a definition of $R_{\alpha\beta}$ and $U_{\beta\gamma\delta}$ the spin connection of this tensorial superspace is SL(n)-valued.

Again one can now perform the field redefinitions (4.1) to end with the OSp$(1|n)$ geometry. As in the case of the conformally flat superspace the field W disappears from the transformed torsion and curvature and eventually we can impose on Φ the linear scalar superfield equation[9]

$$\left(\nabla_{[\alpha} \nabla_{\beta]} + i\frac{\varsigma}{4} C_{\alpha\beta} \right) \Phi(X, \theta) = 0\,.$$

As in the case of the flat tensorial superspace a problem for future study is to understand if the integrability conditions discussed in appendix B allow for the existence of more general, non-linear scalar superfield equation on OSp$(1|n)$ in the form

$$\nabla_{[\alpha} \nabla_{\beta]}\, \Phi = \mathcal{X}_{\alpha\beta}(\Phi)\,. \tag{6.14}$$

[9]Note that, as in the superconformally flat case this equation can be obtained from eq. (6.12) by a formal trick, namely by putting in (6.12) $R_{\alpha\beta} = -\frac{\varsigma}{2} C_{\alpha\beta}\, exp\{(1 + 4/n)\, W/2\}$ and making in the resulting equation

$$\nabla_{[\alpha} \nabla_{\beta]} W + \frac{1}{2} \nabla_\alpha W\, \nabla_\beta W = -\frac{i\varsigma}{2} C_{\alpha\beta} \left(1 - e^{-\frac{W}{2}} \right)$$

the field redefinition $W = 2\, \ln(\frac{\Phi + a}{a})$, $a > 0$.

From the above discussion we can conclude that to construct non-linear higher spin equations involving in a non-trivial way higher spin field potentials one should have at his disposal more general tensorial superspaces than superconformally flat manifolds or manifolds conformally related to $\mathrm{OSp}(1|n)$. However, as we shall demonstrate in the next section, the superconformal tensorial superspaces are the general solution of the supergravity constraints (at least locally, or when the first cohomology of the tensorial superspace is trivial).

7. The general solution of the tensorial supergravity constraints

Let us show that (up to topological subtleties) the superconformally flat and OSp related geometries studied in sections 5 and 6 form the general solution of the tensorial supergravity constraints (3.27), (3.28) and (3.29). To this end consider a weak deviation of tensorial supergeometry from the 'vacuum' solutions, namely, from the flat superspace (3.54), and from the supergroup manifold $\mathrm{OSp}(1|n)$ (3.55).

In the weak superfield approximation over the flat superspace the main superfields describing such a curved tensorial superspace are considered to be infinitesimal of order one, $R_{\alpha\beta} = o(1)$ and $U_{\alpha\,\beta\gamma} = o(1)$. The constraints (3.31), (3.32) and (3.33) on these superfields should be satisfied order by order and in particular in the linear approximation for the infinitesimal quantities of order one. In this approximation we ignore the connection terms in the covariant derivatives (which thus become those of the flat superspace) and drop the second order terms in eq. (3.33). Then eq. (3.33) takes the form

$$\partial_{\alpha\beta} U_{\gamma\delta\sigma} - \partial_{\delta\sigma} U_{\gamma\alpha\beta} = 0\,. \tag{7.1}$$

As a consequence of the Poincaré lemma its general solution is

$$U_{\gamma\alpha\beta} = -\partial_{\alpha\beta}\Psi_\gamma\,. \tag{7.2}$$

Now in the linear approximation eq. (3.31) reduce to

$$\partial_{\gamma\delta} R_{\alpha\beta} = D_{[\alpha|}\partial_{\gamma\delta}\Psi_{|\beta]} = \partial_{\gamma\delta} D_{[\alpha}\Psi_{\beta]}\,. \tag{7.3}$$

Its general solution is

$$R_{\alpha\beta} = D_{[\alpha}\Psi_{\beta]} + a_{\alpha\beta}\,, \qquad \partial_{\gamma\delta} a_{\alpha\beta} = 0\,, \qquad a_{\alpha\beta} = -a_{\beta\alpha} = o(1)\,, \tag{7.4}$$

where $a_{\alpha\beta}$ is independent of $X^{\alpha\beta}$. In the simplest case when $a_{\alpha\beta}$ is a constant matrix it can be absorbed by Ψ_β if one performs the following redefinition $\Psi_\beta \to \Psi_\beta + \theta^\gamma a_{\gamma\beta}$. If $a_{\alpha\beta}$ is a generic polynomial in θ, the solution (7.4) breaks the $\mathrm{GL}(n)$ symmetry and supersymmetry of the original system of supergravity constraints.

When $a_{\alpha\beta} = 0$, eqs. (7.1) and (7.4) describe a weak superfield approximation of the superconformally flat geometry (6.9) and (6.10) with $W_\alpha = \Psi_\alpha = o(1)$. Extending the above analysis to higher orders in the superfields we find that the superconformally flat geometry is the general solution of the constraints on tensorial supergravity which is continuously related to the flat superspace vacuum.

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Let us now consider curved tensorial superspaces with the holonomy group GL(n) or SL(n) whose geometry weakly differs from the 'vacuum' superspace OSp($1|n$) (3.55). In the weak field approximation the superfield $U_{\alpha\beta\gamma}$ is infinitesimal of order one $U_{\alpha\beta\gamma}=o(1)$, while $R_{\alpha\beta}=-\frac{\varsigma}{2}C_{\alpha\beta}+r_{\alpha\beta}$ is of order zero, with $r_{\alpha\beta}$ being infinitesimal of order one . Note also that the covariant derivatives of $R_{\alpha\beta}$ are infinitesimal of order one $\mathcal{D}R_{\alpha\beta}=o(1)$[10]. More explicitly, in the linear approximation

$$\mathcal{D}R_{\alpha\beta}=(\nabla+\Omega)R_{\alpha\beta}=-\frac{\varsigma}{2}\mathcal{D}\,C_{\alpha\beta}+\nabla\,r_{\alpha\beta}=\varsigma\,\Omega_{[\alpha}{}^{\gamma}C_{\beta]\gamma}+\nabla\,r_{\alpha\beta}=o(1)\,, \tag{7.5}$$

where ∇ are the covariant derivatives satisfying the $osp(1|n)$ superalgebra (1.11) and (2.10) (note that $\nabla\,C_{\alpha\beta}=0$), and $\Omega_\alpha{}^\beta$ is an order one deviation of the GL(n) (or SL(n)) connection of the curved superspace from the Sp(n) connection of the supermanifold OSp($1|n$).

In the linear approximation the equation (3.33) takes the form

$$\nabla_{\alpha\beta}U_{\gamma\delta\sigma}-\nabla_{\delta\sigma}U_{\gamma\alpha\beta}-\varsigma\,U_{\gamma\alpha(\sigma}C_{\delta)\beta}-\varsigma\,U_{\gamma\beta(\sigma}C_{\delta)\alpha}=0\,. \tag{7.6}$$

Its general solution is

$$U_{\gamma\alpha\beta}=-\nabla_{\alpha\beta}\Psi_\gamma\,. \tag{7.7}$$

Using (7.7) and (4.3) in the weak superfield approximation one finds that eq. (3.31) reduces to

$$\mathcal{D}_{\gamma\delta}(R_{\alpha\beta}-\mathcal{D}_{[\alpha}\Psi_{\beta]})=-2R_{[\alpha|(\gamma}\,\mathcal{D}_{\delta)}\Psi_{|\beta]}\,, \tag{7.8}$$

and in view of (7.5)

$$\varsigma\,\Omega_{\gamma\delta\,[\alpha}{}^{\epsilon}C_{\beta]\epsilon}-\varsigma\,C_{[\alpha|(\gamma}\,\nabla_{\delta)}\Psi_{|\beta]}=-\nabla_{\gamma\delta}(r_{\alpha\beta}-\nabla_{[\alpha}\Psi_{\beta]})\,. \tag{7.9}$$

One easily sees that a particular solution of (7.9) is

$$\begin{aligned} r_{\alpha\beta}&=\nabla_{[\alpha}\Psi_{\beta]}\quad\rightarrow\quad R_{\alpha\beta}=-\frac{\varsigma}{2}\,C_{\alpha\beta}+\nabla_{[\alpha}\Psi_{\beta]}\,,\\ \Omega_\alpha^{\ \beta}&=(\Omega_{_{OSp}})_\alpha^{\ \beta}+iE^\alpha_{_{OSp}}\,\Psi_\beta-E^{\alpha\gamma}_{_{OSp}}\nabla_\gamma\,\Psi_\beta\,. \end{aligned} \tag{7.10}$$

The solutions (7.7) and (7.10) are the linearized version of (5.1), (5.10) and (5.11) which describe the tensorial superspaces conformally related to the supermanifold OSp($1|n$).

To understand whether a more general solution of the equation (7.9) exists notice that the main superfield $U_{\alpha\beta\gamma}$ expressed as in (7.6) can be put to zero by using generalized super-Weyl transformations (5.1) with $W_\alpha=-\Psi_\alpha$. Equivalently, one can simply put $\Psi_\alpha=0$ in eqs. (7.6), (7.8) and (7.9). We thus exclude from further consideration the superconformal solution already found above. Then (7.8) and (7.9) reduce to

$$\mathcal{D}_{\gamma\delta}R_{\alpha\beta}=0\quad\rightarrow\quad\varsigma\,\Omega_{\gamma\delta\,[\alpha}{}^{\epsilon}C_{\beta]\epsilon}=-\nabla_{\gamma\delta}r_{\alpha\beta}\,. \tag{7.11}$$

[10]We should stress that one is not obliged to use the explicit form of the OSp-vacuum solution involving the constant matrix $C_{\alpha\beta}$ which breaks manifest GL(n) or SL(n) gauge invariance. All the consideration can be carried out in a GL(n) covariant fashion by using two properties of the 'near-OSp' superfields: $R_{\alpha\beta}$ should be a non-degenerate matrix of order zero and $\mathcal{D}R_{\alpha\beta}$ is infinitesimal of order one.

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If we restrict the consideration to superspaces of SL(n) holonomy then $R_{\alpha\beta}$ must also satisfy the condition

$$\mathcal{D}_{(\gamma}\, R_{\beta)\alpha} = 0\,, \tag{7.12}$$

which follows from the fact that for the spaces of SL(n) holonomy $U_{\alpha\beta\gamma} = G_{\alpha\beta\gamma} + \frac{2i}{3}\mathcal{D}_{(\gamma}\, R_{\beta)\alpha}$ (see subsection 3.4) and in the case under consideration $U_{\alpha\beta\gamma} = 0$.

It can be shown, using the commutation relations (4.3) for the covariant derivatives and assuming $R_{\alpha\beta}$ to have the inverse matrix, that a stronger condition holds $\mathcal{D}_\gamma\, R_{\beta\alpha} = 0$. Then together with (7.11) this implies that $R_{\alpha\beta}$ is covariantly constant $\mathcal{D}\, R_{\alpha\beta} = 0$[11] whose integrability (in view of the constraints (3.27)–(3.29) on the torsion and curvature) forces the tensorial superspace to be the supergroup manifold OSp($1|n$).

We have thus shown that the general solution of the tensorial supergravity constraints are the superspaces conformally related to flat superspace or supergroup manifold OSp($1|n$).

8. Conclusion and discussion

The main results of this article are the following

- we have found simple free equations of motion of a scalar superfield propagating in flat tensorial superspace (2.3) and in the supergroup manifold OSp($1|n$) (2.11) which in the case of $n = 4$ describe the infinite set of OSp($1|8$) invariant free higher spin field equations in flat $D = 4$ and AdS_4 space-time, respectively; in the cases of $n = 8$ and $n = 16$, which correspond to $D = 6$ and $D = 10$ space-time, these equations describe conformally invariant higher spin fields with self-dual field strengths (work in progress);

- the geometry of curved tensorial superspaces has been introduced and corresponding supergravity constraints have been obtained from the requirement of the κ-symmetry of superparticle dynamics in the tensorial supergravity background; the superfield structure of the tensorial supergravity has been shown to be a generalization of $N = 1$, $D = 3$ supergravity;

- A 'no-go' result is that the class of the superconformally flat and OSp($1|n$)-related superspaces is the general solution of the constraints of tensorial supergravity with GL(n) or SL(n) holonomy which are required by the κ-symmetry of the GL(n)-*invariant* tensorial superparticle.

  As we have shown, the geometry of these superspaces is trivial in the sense that it cannot produce 'minimal-like' interactions of higher spin fields.

[11] Indeed, (7.12), (7.11) and (4.3) with $U_{\beta\;\gamma\delta} = 0$ imply $\mathcal{D}R_{\alpha(\gamma}\, R_{\delta)\beta} + \mathcal{D}R_{\beta(\gamma}\, R_{\delta)\alpha} = 0$ which also can be written in the form $\mathcal{D}(R_{\alpha(\gamma}\, R_{\delta)\beta}) = 0$. In the case with an invertible $R_{\alpha\beta}$ one multiplies this equation by $R^{-1\,\epsilon\alpha}R^{-1\,\kappa\beta}$ to arrive at $(R^{-1}\mathcal{D}R)_{(\gamma}{}^{(\epsilon}\,\delta_{\delta)}{}^{\kappa)} = 0$ which implies $(R^{-1}\mathcal{D}R)_\gamma{}^\epsilon = 0$ and, hence, $\mathcal{D}R_{\beta\gamma} = 0$.

During work on this project we have also analyzed the possibility of constructing a tensorial super-Yang-Mills theory and its coupling to the scalar superfield and have not found a nontrivial model of this kind which possess manifest $GL(n)$ or $SL(n)$ symmetry and non-manifest $OSp(1|2n)$ generalized superconformal symmetry. One can thus assume that, surprisingly enough, the scalar superfield is the only dynamical object in the tensorial superspace of this kind. This is similar to the unfolded higher spin field dynamics of Vasiliev where at the linearized level all physical degrees are contained in a scalar field (zero form). Since interactions of higher spin fields break conformal invariance one should look for tensorial superfield models in which the generalized superconformal group $OSp(1|2n)$ and a corresponding structure group $GL(n)$ or $SL(n)$ are (spontaneously) broken down to an appropriate subgroup (or realized non-linearly with an appropriate linearly realized subgroup). The unbroken/linearly realized subgroup of $GL(n)$ or $SL(n)$ should presumably be the Lorentz group $SO(1, D-1)$ of the associated D-dimensional subspace-time of the tensorial superspace.

One might hope that in such models the tensorial supergravity constraints are less restrictive.

Note that in the unfolded formulation of non-linear higher spin dynamics conformal symmetry is spontaneously broken by doubling auxiliary (spinor or vector) variables and introducing Goldstone-like fields which acquire non-zero vacuum expectation values [5].

Our results suggest that for tensorial supergravity to be a relevant geometrical framework for the formulation of non-linear dynamics of higher spin fields in a way which would be somewhat alternative to the unfolded higher spin dynamics [27, 10, 1] one should enlarge the superspace with additional coordinates, for example, by keeping in the non-linear construction the auxiliary commuting spinor variables which were used to construct the superparticle action in the tensorial superspace and which entered the 'preonic' field equations (1.4) and (1.10). In this respect let us conclude with the following comment. As we have already mentioned, most of the known approaches to the description of higher spin theories use additional variables, like vector variables [5] or bosonic spinor variables (see e.g. [27, 10, 1, 39] and refs. therein). The construction of non-linear higher spin equations based on the unfolded formulations requires the doubling of the auxiliary variables *of the same kind* [5] and (spontaneous) breaking of conformal symmetry.

When higher spin theories are formulated in a tensorial space or superspace, as discussed in this paper, in addition to the ordinary space-time coordinates x^m one introduces auxiliary tensorial variables ($y^{mn} = -y^{nm}$ for $D = 4$). Higher spin field equations can be regarded as those which describe the physical states of a first quantized particle. To construct an appropriate classical mechanics of this particle one also needs bosonic spinor variables λ_α. The quantization of this particle mechanics [17] produces the field equation (1.4) or (1.5). Then, in the free field theory case one can consistently eliminate the dependence of the wave functions on either the tensorial variables y^{mn} and recover the unfolded formulation [21, 24, 25], or on the spinorial variables λ_α and get the higher spin field equations (1.6) in tensorial spaces [21, 22]. Thus, in view of the above remark on 'doubling' one can assume that the formulation of the non-linear dynamics of higher spin

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fields in the framework of tensorial SYM or supergravity may require both the tensorial and spinorial auxiliary variables. In this perspective the superfield generalization of the 'preonic' equations (2.8) and (2.16) may play a special role.

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A. Spinor superfield equations

One may ask whether it is possible instead of considering the scalar superfield equations (2.3) to incorporate component eqs. (1.6) and (1.7) into equations for a spinor superfield whose leading component is the fermionic field $f_\alpha(X)$? The answer to this question is positive, although one should require the spinor superfield to obey a set of two equations

$$D_{[\alpha}\Psi_{\beta]}(X,\theta) = 0\,, \tag{A.1}$$
$$\partial_{\alpha[\beta}\Psi_{\gamma]}(X,\theta) = 0\,. \tag{A.2}$$

Indeed, in virtue of eq. (A.1), one finds that $D_{[\beta}D_{\gamma]}\Psi_\alpha = 2i\partial_{\alpha[\beta}\Psi_{\gamma]}$. Then, because of (A.2),

$$D_{[\alpha}D_{\beta]}\Psi_\alpha(X,\theta) = 0\,. \tag{A.3}$$

Eq. (A.3) implies that the spinor superfield $\Psi_\alpha(X,\theta)$ contains only two non-zero components

$$\Psi_\alpha(X,\theta) = f_\alpha(X) + \theta^\beta k_{\beta\alpha}(X)\,. \tag{A.4}$$

Imposing eq. (A.1), one finds that the bosonic spin-tensor $k_{\beta\alpha}$ is symmetric, $k_{\beta\alpha} = k_{\alpha\beta}$, and the fermionic field $f_\alpha(X)$ obeys the equations (1.7). The same equations follow from eq. (A.2), which also implies that $\partial_{\alpha[\beta}k_{\gamma]\delta} = 0$. The latter can be decomposed into

$$\partial_{\alpha[\beta}k_{\gamma]\delta} + \partial_{\delta[\beta}k_{\gamma]\alpha} = 0\,, \tag{A.5}$$
$$\partial_{\alpha[\beta}k_{\gamma]\delta} - \partial_{\delta[\beta}k_{\gamma]\alpha} = 0\,. \tag{A.6}$$

Eqs. (A.5) are actually a kind of Bianchi identities which imply that the symmetric spin tensor $k_{\alpha\beta}$ is the derivative of a scalar field

$$k_{\alpha\beta} = \partial_{\alpha\beta}b(X)\,. \tag{A.7}$$

Then eqs. (A.6) and (A.7) reduce to the equation (1.6) for the scalar field $b(X)$.

On the other hand, the form of the superfield (A.4) with $k_{\alpha\beta} = \partial_{\alpha\beta} b(X)$ implies that Ψ_α is the derivative of a scalar superfield Φ obeying eqs. (2.3),

$$i\Psi_\alpha(X,\theta) = D_\alpha \Phi(X,\theta)\,, \qquad D_{[\alpha}D_{\beta]}\Phi(X,\theta) = 0\,. \tag{A.8}$$

Eqs. (A.8) provide the general solution of eqs. (A.1) and (A.2). Thus both, the scalar and spinor superfeild representation of the system of the free higher spin equations (1.6), (1.7) are completely equivalent.

B. A generic form of the scalar superfield equation in a supergravity background

Consider the equation

$$\mathcal{D}_{[\beta}\mathcal{D}_{\gamma]}\,\Phi = \frac{i}{2}\,\mathcal{X}_{\beta\gamma}\,, \tag{B.1}$$

where $\mathcal{X}_{\beta\gamma}(Z)$ is an antisymmetric tensor superfield. In section 4 we dealt with $\mathcal{X}_{\beta\gamma} = R_{\alpha\beta}\,\Phi$, and now we shall consider the case of a generic $\mathcal{X}_{\beta\gamma}(Z) = -\mathcal{X}_{\gamma\beta}(Z)$.

Acting on (B.1) with $\mathcal{D}_\alpha$ we arrive at a more general form of the equation (4.5)

$$\mathcal{D}_{\alpha[\beta}\mathcal{D}_{\gamma]}\Phi = \frac{1}{3}R_{\beta\gamma}\,\mathcal{D}_\alpha\Phi - \frac{1}{3}R_{\alpha[\beta}\,\mathcal{D}_{\gamma]}\Phi + \frac{1}{6}\mathcal{D}_\alpha\mathcal{X}_{\beta\gamma} - \frac{1}{6}\mathcal{D}_{[\beta}\mathcal{X}_{\gamma]\alpha}\,, \tag{B.2}$$

Then acting on (B.2) with $\mathcal{D}_\delta$ we get a generalization of the equation (4.6)

$$\begin{aligned}\mathcal{D}_{\alpha[\beta}\mathcal{D}_{\gamma]\delta}\Phi = {} & \frac{1}{2}\mathcal{D}_{\alpha[\beta}\mathcal{X}_{\gamma]\delta} - \frac{i}{6}\mathcal{D}_\delta\mathcal{D}_\alpha\mathcal{X}_{\beta\gamma} + \frac{i}{6}\mathcal{D}_\delta\mathcal{D}_{[\beta}\mathcal{X}_{\gamma]\alpha} + \frac{1}{2}R_{\alpha\delta}\mathcal{X}_{\beta\gamma} + \\ & + U_{[\beta\ \gamma]\alpha}\,\mathcal{D}_\delta\Phi + \frac{i}{2}F_{\delta\,\alpha[\beta}\mathcal{D}_{\gamma]}\Phi + \frac{i}{6}\mathcal{D}_\delta R_{\beta\gamma}\,\mathcal{D}_\alpha\Phi - \frac{i}{6}\mathcal{D}_\delta R_{\alpha[\beta}\,\mathcal{D}_{\gamma]}\Phi + \\ & + i\,\mathcal{D}_\alpha\mathcal{D}_{[\beta}\Phi\,R_{\gamma]\delta} - \frac{i}{3}R_{\beta\gamma}\,\mathcal{D}_\delta\mathcal{D}_\alpha\Phi + \frac{i}{3}\mathcal{D}_\delta\mathcal{D}_{[\beta}\Phi\,R_{\beta]\alpha}\,.\end{aligned} \tag{B.3}$$

The integrability condition (4.7) of eq. (B.3) imposes the following restriction on the form of $\mathcal{X}_{\alpha\beta}$

$$\begin{aligned}& 3\mathcal{D}_{(\alpha[\beta}\mathcal{X}_{\gamma]\delta)} + \mathcal{D}_{\alpha\delta}\mathcal{X}_{\beta\gamma} + i\mathcal{D}_{(\alpha}\mathcal{D}_{[\beta}\mathcal{X}_{\gamma]\delta)} + 4R_{(\alpha[\beta}\mathcal{X}_{\gamma]\delta)} = \\ & \quad = i\mathcal{D}_{(\alpha}R_{\delta)[\beta}\mathcal{D}_{\gamma]}\Phi - i\mathcal{D}_{[\beta}R_{\gamma](\alpha}\mathcal{D}_{\delta)}\Phi + R_{\beta\gamma}\mathcal{D}_{\alpha\delta}\Phi + 2R_{(\alpha[\beta}\mathcal{D}_{\gamma]\delta)}\Phi\,.\end{aligned} \tag{B.4}$$

One of the solutions of (B.4) considered in section 4 is $\mathcal{X}_{\alpha\beta} = R_{\alpha\beta}\Phi$.

As we discuss in appendix C, in the case of $n = 2$ which corresponds to $N = 1$, $D = 3$ supergravity coupled to a scalar superfield the integrability condition (4.7) valid for a generic R and U satisfying the off-shell SL(2) holonomy constraints allows to choose $\mathcal{X}_{\alpha\beta} = \epsilon_{\alpha\beta}\,f(Z)$, and in particular $\mathcal{X}_{\alpha\beta} = 0$ or $\mathcal{X}_{\alpha\beta} = m\epsilon_{\alpha\beta}$, where m is a constant of the dimension of mass.

In the case of a generic $n > 2$ the scalar field equation (B.1) with $\mathcal{X}_{\alpha\beta} = 0$ is satisfied if the right hand side of (B.4) vanishes. A particular solution of this constraint is when

$$\Phi = f(W)\,, \qquad \mathcal{D}_{[\alpha}\,\mathcal{D}_{\beta]}\Phi = \mathcal{D}_{[\alpha}\,\mathcal{D}_{\beta]}f(W) = 0 \tag{B.5}$$

and the superspace is superconformally flat (6.9)–(6.10) (or equivalently (6.1)–(6.8)) such that, in virtue of (B.5),

$$R_{\alpha\beta} = i\left(\frac{1}{2} + \frac{f''}{f'}\right)\mathcal{D}_\alpha W\,\mathcal{D}_\beta W, \quad f'(W) \equiv \frac{df(W)}{dW}\,. \tag{B.6}$$

In the basis of the flat covariant derivatives the scalar superfield equation takes the form

$$D_{[\alpha}D_{\beta]}\,W = -\left(1 + \frac{f''}{f'}\right) D_\alpha W\, D_\beta W\,. \tag{B.7}$$

Upon the following field redefinition $\tilde{\Phi}(W) = c \cdot \int e^W df(W)$, it reduces to the free scalar superfield equation

$$D_{[\alpha}D_{\beta]}\,\tilde{\Phi} = 0\,. \tag{B.8}$$

A question is whether for $n > 2$ there exist a scalar superfield equation with $\mathcal{X}_{\alpha\beta}(Z) \neq R_{\alpha\beta}\Phi$ whose integrability conditions do not reduce the tensorial superspace to the (superconformally) flat or $\mathrm{OSp}(1|n)$ supermonifold and therefore do not trivialize it.

C. Peculiarities of $N = 1$, $D = 3$ supergravity

Since the $N = 1$, $D = 3$ supergravity is a particular example of our generic construction, before concluding the paper let us briefly discuss this well known case from our perspective.

In this case the holonomy group is $\mathrm{SL}(2) \sim Sp(2)$, and the antisymmetric tensors are proportional to $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$ ($\epsilon_{12} = 1$), for instance $R_{\alpha\beta}(Z) = \epsilon_{\alpha\beta}R(Z)$. As one can check (see appendix B for details), in addition to eq. (4.1) such a simplification allows for other, well known, forms of the scalar superfield equations coupled to the off-shell $N = 1$, $D = 3$ supergravity satisfying the constraints (3.52) and (3.53), namely, the massless superfield equation

$$\mathcal{D}_{[\alpha}\mathcal{D}_{\beta]}\,\Phi = \frac{1}{2}\epsilon_{\alpha\beta}\,\epsilon^{\gamma\delta}\,\mathcal{D}_\gamma\mathcal{D}_\delta\Phi = 0 \qquad \Rightarrow \quad \mathcal{D}^\alpha\mathcal{D}_\alpha\Phi = 0\,, \tag{C.1}$$

and the massive superfield equation

$$\mathcal{D}_{[\alpha}\mathcal{D}_{\beta]}\,\Phi - \frac{im}{2}\epsilon_{\alpha\beta}\,\Phi = \epsilon_{\alpha\beta}(\mathcal{D}^\gamma\mathcal{D}_\gamma\Phi - im\Phi) = 0 \quad \Rightarrow \quad \mathcal{D}^\alpha\mathcal{D}_\alpha\Phi = im\Phi\,. \tag{C.2}$$

Moreover, in the case of $N = 1$, $D = 3$ superspace the non-linear equation of the scalar superfield has the following general form

$$\mathcal{D}^\alpha\mathcal{D}_\alpha\Phi = if(Z)\,\Phi\,. \tag{C.3}$$

with $f(Z)$ being an arbitrary superfield.

Since on the mass shell $D = 3$ supergravity is completely determined by its coupling to the matter fields, we can assume $R(Z)$ to be a function of $\Phi(Z)$, i.e. $R = R(\Phi(Z))$. Then (C.1)–(C.3) describe a non-linear self-interaction of the scalar superfield $\Phi(Z)$.

The equations (C.1)–(C.3) are compatible both with Poincaré and AdS $N=1$, $D=3$ supergravity. However, this is not the case for tensorial supergravity with a generic n, in which case, for example, the Sp(n) holonomy required by (C.1)–(C.3) reduces the tensorial supergravity down to the supergroup manifold OSp($1|n$), since $\mathcal{D}C_{\alpha\beta}=0$.

Let us note that in the case of $N=1$, $D=3$ supergravity the equations (6.7)–(6.10), (6.11) and (6.14) (with a generic function of W (or Φ) on the right hand side), or equivalently eqs. (C.3), are Lagrangian in the sense that they can be derived from the $N=1$, $D=3$ supergravity action [30] coupled to a scalar field

$$S=\frac{1}{2}\int d^3x d^2\theta \,\mathrm{sdet}\, E_B{}^A \Big[R+\epsilon^{\alpha\beta}\mathcal{D}_\alpha\Phi\mathcal{D}_\beta\Phi+\varsigma\cdot\mathcal{X}(\Phi)\Big]\,. \qquad \text{(C.4)}$$

In the generic case of $n>2$ it is still an open problem to figure out whether the equations (6.7)–(6.10), (6.11) and (6.14), as well as (2.3) and (2.11) can be obtained by the variation of a corresponding action.

References

[1] M.A. Vasiliev, *Progress in higher spin gauge theories*, hep-th/0104246; *Higher-spin theories and Sp(2m) invariant space-time*, hep-th/0301235, and refs. therein.

[2] D. Sorokin, *Introduction to the classical theory of higher spins*, hep-th/0405069.

[3] E.S. Fradkin and M.A. Vasiliev, *On the gravitational interaction of massless higher spin fields*, *Phys. Lett.* **B 189** (1987) 89; *Cubic interaction in extended theories of massless higher spin fields*, *Nucl. Phys.* **B 291** (1987) 141.

[4] M.A. Vasiliev, *Consistent equations for interacting massless fields of all spins in the first order in curvatures*, *Ann. Phys. (NY)* **190** (1989) 59.

[5] M.A. Vasiliev, *Nonlinear equations for symmetric massless higher spin fields in (a)ds(d)*, *Phys. Lett.* **B 567** (2003) 139 [hep-th/0304049]; *Higher spin gauge theories in various dimensions*, *Fortschr. Phys.* **52** (2004) 702 [hep-th/0401177].

[6] A.K.H. Bengtsson, I. Bengtsson and L. Brink, *Cubic interaction terms for arbitrary spin*, *Nucl. Phys.* **B 227** (1983) 31; *Cubic interaction terms for arbitrarily extended supermultiplets*, *Nucl. Phys.* **B 227** (1983) 41.

[7] F.A. Berends, G.J.H. Burgers and H. Van Dam, *On spin three selfinteractions*, *Z. Physik* **C 24** (1984) 247; *On the theoretical problems in constructing interactions involving higher spin massless particles*, *Nucl. Phys.* **B 260** (1985) 295.

[8] T. Damour and S. Deser, *Higher derivative interactions of higher spin gauge fields*, *Class. and Quant. Grav.* **4** (1987) L95.

[9] M.A. Vasiliev, *Unfolded representation for relativistic equations in (2+1) anti-de Sitter space*, *Class. and Quant. Grav.* **11** (1994) 649.

[10] M.A. Vasiliev, *Higher spin gauge theories: star-product and AdS space*, hep-th/9910096.

[11] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142].

[12] I.L. Buchbinder, A. Pashnev and M. Tsulaia, *Lagrangian formulation of the massless higher integer spin fields in the AdS background*, *Phys. Lett.* **B 523** (2001) 338 [hep-th/0109067]; *Massless higher spin fields in the AdS background and brst constructions for nonlinear algebras*, hep-th/0206026;
G. Bonelli, *On the tensionless limit of bosonic strings, infinite symmetries and higher spins*, *Nucl. Phys.* **B 669** (2003) 159 [hep-th/0305155]; *On the covariant quantization of tensionless bosonic strings in AdS spacetime*, *J. High Energy Phys.* **11** (2003) 028 [hep-th/0309222]; *On the boundary gauge dual of closed tensionless free strings in AdS*, hep-th/0407144;
A. Sagnotti and M. Tsulaia, *On higher spins and the tensionless limit of string theory*, *Nucl. Phys.* **B 682** (2004) 83 [hep-th/0311257];
X. Bekaert, I.L. Buchbinder, A. Pashnev and M. Tsulaia, *On higher spin theory: strings, brst, dimensional reductions*, *Class. and Quant. Grav.* **21** (2004) S1457 [hep-th/0312252].

[13] I.A. Bandos and A.A. Zheltukhin, *Null super p-branes quantum theory in four-dimensional space-time*, *Fortschr. Phys.* **41** (1993) 619 and refs. therein.

[14] C. Fronsdal, *Massless particles, ortosymplectic symmetry and another type of Kaluza-Klein theory*, Preprint UCLA/85/TEP/10, in *Essays on supersymmetry*, Reidel, 1986, *Mathematical physics studies*, v. 8.

[15] I.A. Bandos and J. Lukierski, *Tensorial central charges and new superparticle models with fundamental spinor coordinates*, *Mod. Phys. Lett.* **A 14** (1999) 1257 [hep-th/9811022].

[16] I.A. Bandos and J. Lukierski, *New superparticle models outside the hls supersymmetry scheme*, *Lect. Notes Phys.* **539** (2000) 195–210 [hep-th/9812074].

[17] I.A. Bandos, J. Lukierski and D.P. Sorokin, *Superparticle models with tensorial central charges*, *Phys. Rev.* **D 61** (2000) 045002 [hep-th/9904109].

[18] I.A. Bandos, J.A. de Azcárraga, J.M. Izquierdo, M. Picón and O. Varela, *On BPS preons, generalized holonomies and $D = 11$ supergravities*, *Phys. Rev.* **D 69** (2004) 105010 [hep-th/0312266].

[19] I.A. Bandos, J.A. de Azcárraga, J.M. Izquierdo and J. Lukierski, *BPS states in M-theory and twistorial constituents*, *Phys. Rev. Lett.* **86** (2001) 4451 [hep-th/0101113].

[20] I.A. Bandos, J. Lukierski, C. Preitschopf and D.P. Sorokin, *Osp supergroup manifolds, superparticles and supertwistors*, *Phys. Rev.* **D 61** (2000) 065009 [hep-th/9907113].

[21] M.A. Vasiliev, *Conformal higher spin symmetries of 4d massless supermultiplets and* $\mathrm{OSp}(l, 2m)$ *invariant equations in generalized (super)space*, *Phys. Rev.* **D 66** (2002) 066006 [hep-th/0106149].

[22] M.A. Vasiliev, *Relativity, causality, locality, quantization and duality in the* $\mathrm{Sp}(2m)$ *invariant generalized space-time*, hep-th/0111119.

[23] V.E. Didenko and M.A. Vasiliev, *Free field dynamics in the generalized AdS (super)space*, *J. Math. Phys.* **45** (2004) 197 [hep-th/0301054].

[24] M. Plyushchay, D. Sorokin and M. Tsulaia, *Higher spins from tensorial charges and* $\mathrm{OSp}(n|2n)$ *symmetry*, *J. High Energy Phys.* **04** (2003) 013 [hep-th/0301067].

[25] M. Plyushchay, D. Sorokin and M. Tsulaia, *GL flatness of* $\mathrm{OSp}(1|2n)$ *and higher spin field theory from dynamics in tensorial spaces*, hep-th/0310297.

[26] O.A. Gelfond and M.A. Vasiliev, *Higher rank conformal fields in the* Sp($2m$) *symmetric generalized space-time*, hep-th/0304020.

[27] M.A. Vasiliev, *Equations of motion of interacting massless fields of all spins as a free differential algebra*, *Phys. Lett.* **B 209** (1988) 491; *Consistent equations for interacting massless fields of all spins in the first order in curvatures*, *Ann. Phys. (NY)* **190** (1989) 59.

[28] M.A. Vasiliev, *Extended higher spin superalgebras and their realizations in terms of quantum operators*, *Fortschr. Phys.* **36** (1988) 33;
S.E. Konstein and M.A. Vasiliev, *Extended higher spin superalgebras and their massless representations*, *Nucl. Phys.* **B 331** (1990) 475.

[29] J. Engquist, E. Sezgin and P. Sundell, *On* $N = 1, 2, 4$ *higher spin gauge theories in four dimensions*, *Class. and Quant. Grav.* **19** (2002) 6175 [hep-th/0207101]; *Superspace formulation of 4D higher spin gauge theory*, *Nucl. Phys.* **B 664** (2003) 439 [hep-th/0211113].

[30] M. Brown and S.J. Gates Jr., *Superspace Bianchi identities and the supercovariant derivative*, *Ann. Phys. (NY)* **122** (1979) 443;
B.M. Zupnik and D.G. Pak, *Superfiled formulation of simplest three-dimansional gauge theories and conformal supergravities*, *Theor. Math. Phys.* **77** (1988) 1070; *Differential and integral forms in supergrauge theories and supergravity*, *Class. and Quant. Grav.* **6** (1989) 723;
M. Becker et al., *M-theory on* Spin(7) *manifolds, fluxes and 3d,* $N = 1$ *supergravity*, *Nucl. Phys.* **B 683** (2004) 67 [hep-th/0312040].

[31] E.S. Fradkin and M.A. Vasiliev, *Superalgebra of higher spins and auxiliary fields*, *Int. J. Mod. Phys.* **A 3** (1988) 2983;
S.E. Konstein and M.A. Vasiliev, *Extended higher spin superalgebras and their massless representations*, *Nucl. Phys.* **B 331** (1990) 475.

[32] N. Dragon, *Torsion and curvature in extended supergravity*, *Z. Physik* **C 2** (1979) 29.

[33] S.J. Gates Jr. and W. Siegel, *Understanding constraints in superspace formulations of supergravity*, *Nucl. Phys.* **B 163** (1980) 519;
S.J. Gates Jr., K.S. Stelle and P.C. West, *Algebraic origins of superspace constraints in supergravity*, *Nucl. Phys.* **B 169** (1980) 347.

[34] P.S. Howe and G. Papadopoulos, $N = 2$*,* $D = 2$ *supergeometry*, *Class. and Quant. Grav.* **4** (1987) 11;
P.S. Howe, *Weyl superspace*, *Phys. Lett.* **B 415** (1997) 149 [hep-th/9707184].

[35] I.A. Bandos, E. Ivanov, J. Lukierski and D. Sorokin, *On the superconformal flatness of AdS superspaces*, *J. High Energy Phys.* **06** (2002) 040 [hep-th/0205104].

[36] B. de Wit and D.Z. Freedman, *Systematics of higher spin gauge fields*, *Phys. Rev.* **D 21** (1980) 358.

[37] M.T. Grisaru, P.S. Howe, L. Mezincescu, B. Nilsson and P.K. Townsend, $N = 2$ *superstrings in a supergravity background*, *Phys. Lett.* **B 162** (1985) 116;
E. Bergshoeff, E. Sezgin and P.K. Townsend, *Supermembranes and eleven-dimensional supergravity*, *Phys. Lett.* **B 189** (1987) 75;
J.A. Shapiro and C.C. Taylor, *The space-time supersymmetric formulation of the string*, *Phys. Rept.* **191** (1990) 221;
I.A. Bandos and J.M. Isidro, $D = 4$ *supergravity dynamically coupled to superstring in a superfield lagrangian approach*, *Phys. Rev.* **D 69** (2004) 085009 [hep-th/0308102], and references therein.

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[38] S.J. Gates, M.T. Grisaru, M. Roček and W. Siegel, *Superspace, or one thousand and one lessons in supersymmetry*, *Front. Phys.* **58** (1983) 1 [hep-th/0108200];
J. Wess and J. Bagger, *Supersymmetry and supergravity*, Princeton University Press, 1992, p. 259;
I.L. Buchbinder and S.M. Kuzenko, *Ideas and methods of supersymmetry and supergravity or a walk through superspace*, IOP Publishing, Bristol, UK, 1995, p. 640;
W. Siegel, *Fields*, hep-th/9912205.

[39] K.B. Alkalaev and M.A. Vasiliev, *$N = 1$ supersymmetric theory of higher spin gauge fields in AdS_5 at the cubic level*, *Nucl. Phys.* **B 655** (2003) 57 [hep-th/0206068].